

AF INVERSE MONOIDS AND THE STRUCTURE OF COUNTABLE MV-ALGEBRAS

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ABSTRACT. We define a class of inverse monoids having the property that their lattices of principal ideals naturally form an MV-algebra. We say that an arbitrary MV-algebra can be co-ordinatized if it is isomorphic to one constructed in this way from such a monoid. We prove that every countable MV-algebra can be so co-ordinatized. The particular inverse monoids needed to establish this result are examples of what we term AF inverse monoids. These are constructed from Bratteli diagrams and arise naturally as direct limits of finite direct products of finite symmetric inverse monoids.

1. INTRODUCTION

MV-algebras were introduced by C. C. Chang in 1958 [15]. In Chang's original axiomatization, it is plain that such algebras are generalizations of Boolean algebras. In general, the elements of an MV-algebra are not idempotent, but those that are form a Boolean algebra. A good introduction to their theory may be found in Mundici's tutorial notes [52]. The standard reference is [16]. The starting point for our paper is Mundici's own work that connects countable MV-algebras to a class of AF C^* -algebras [50, 53]. He sets up a correspondence between AF C^* -algebras whose Murray-von Neumann order is a lattice and countable MV-algebras. In [51], he argues that AF algebras 'should be regarded as sort of noncommutative Boolean algebras'. This is persuasive because the commutative AF C^* -algebras are function algebras over separable Boolean spaces. But the qualification 'sort of' is important. The result would be more convincing if commutative meant, precisely, countable Boolean algebra. In this paper, we shall introduce a class of countable structures whose commutative members are precisely this.

Approximately finite (AF) C^* -algebras, that is those C^* -algebras which are direct limits of finite dimensional C^* -algebras, were introduced by Bratteli in 1972 [12], and form one of the most important classes of C^* -algebras. Reading Bratteli's paper, it quickly becomes apparent that his calculations rest significantly on the properties of matrix units. The reader will recall that these are square matrices all of whose entries are zero except for one place where the entry is one. Our key observation is that matrix units of a given size n form a groupoid, and this groupoid determines the structure of a finite symmetric inverse monoid on n letters. The connection is via what are termed rook matrices [61]. Symmetric inverse monoids are simply the monoids of all partial bijections of a given set. Here the set can be taken to be $\{1, \dots, n\}$. These monoids have a strong Boolean character. For

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example, their semilattices of idempotents form a finite Boolean algebra. They are however non-commutative. This leads us to define a general class of Boolean inverse monoids, called AF inverse monoids, constructed from Bratteli diagrams. We argue that this class of monoids is the most direct non-commutative generalization of Boolean algebras. For example, they figure in the developing theory of non-commutative Stone dualities [42, 43, 44, 45, 46] where they are associated with a class of étale topological groupoids. Significantly, commutative AF inverse monoids are countable Boolean algebras. It is worth noting that the groups of units of such inverse monoids have already been studied [17, 35, 37] though sans the inverse monoids.

We prove that the poset of principal ideals of an AF inverse monoid naturally forms an MV-algebra when that poset is a lattice. Accordingly, we say that an MV-algebra that is isomorphic to an MV-algebra constructed in this way may be co-ordinatized by an inverse monoid. The main theorem we prove in this paper is that *every* countable MV-algebra may be co-ordinatized in this way. As a concrete example, we provide an explicit description of the AF inverse monoid that co-ordinatizes the MV-algebra of dyadic rationals in the unit interval. It turns out to be a discrete version of the CAR algebra. Finally, our results can be viewed as contributing to the study of the poset of \mathcal{J} -classes of an inverse semigroup. For results in this area and further references, see [48]. There are also *thematic* links between our work and that to be found in [4, 27, 63]. This has influenced our choice of terminology when referring to partial refinement monoids.

2. BASIC DEFINITIONS

We shall work with two classes of structures in this paper: inverse monoids and partial commutative monoids. In addition to defining the structures we shall be working with, we shall also define precisely what we mean by co-ordinatizing an MV-algebra by means of an inverse monoid.

2.1. Boolean inverse monoids. For inverse semigroup theory, we refer the reader to [38]. However, we need little theory *per se*, rather a number of definitions and some very basic examples. Recall that an *inverse semigroup* is a semigroup S in which for each element s there is a unique element s^{-1} satisfying $s = ss^{-1}s$ and $s^{-1} = s^{-1}ss^{-1}$. Inverse semigroups are well-equipped with idempotents since both $s^{-1}s$ and ss^{-1} are idempotents. The set of idempotents of S is denoted by $E(S)$ and is always a commutative idempotent subsemigroup. For this reason, it is usually referred to as the *semilattice of idempotents* of S . Our inverse semigroups will always have a zero and ultimately an identity and so will be monoids. If a is an element of an inverse semigroup such that $e = a^{-1}a$ and $f = aa^{-1}$, then we shall often write $e \xrightarrow{a} f$ or $e \mathcal{D} f$ and say that e is the *domain* of a and f is the *range* of a . Accordingly, we define $\mathbf{d}(a) = a^{-1}a$ and $\mathbf{r}(a) = aa^{-1}$. We define $s \mathcal{D} t$ if $\mathbf{d}(s) \mathcal{D} \mathbf{d}(t)$, and $s \mathcal{J} t$ if $SsS = StS$. Observe that $\mathcal{D} \subseteq \mathcal{J}$. These are the only two of Green's relations needed in this paper.

Three relations definable on any inverse semigroup will play significant rôles. The *natural partial order* \leq is defined by $a \leq b$ if $a = be$ for some idempotent e . Despite appearances it is ambidextrous, compatible with the multiplication, and $a \leq b$ implies that $a^{-1} \leq b^{-1}$. Observe that if $a, b \leq c$ then both $a^{-1}b$ and ab^{-1} are idempotents. This leads to the definition of our second relation. The *compatibility relation* \sim is defined as follows: $a \sim b$ if $a^{-1}b$ and ab^{-1} are both idempotents. Thus $a \sim b$ is a necessary condition for a and b to have a join. A subset is said to be compatible if any two elements in the subset are compatible. Finally, there is a refinement of the compatibility relation. Elements of an inverse semigroup a and b are said to be *orthogonal*, denoted by $a \perp b$, if $a^{-1}b = 0 = ab^{-1}$.

The particular classes of inverse monoids we shall use in this paper may now be defined. An inverse monoid with zero is said to be *distributive* if its semilattice of idempotents is a distributive lattice, all compatible binary joins exist, and multiplication distributes over binary joins. A distributive inverse monoid is said to be *Boolean* if its lattice of idempotents is a Boolean algebra. Morphisms of distributive inverse monoids are monoid homomorphisms that map zero to zero and which preserve binary compatible joins. More about Boolean and distributive inverse monoids can be found in [44, 45, 46]. An inverse semigroup in which all binary meets exist is called a \wedge -monoid.

The key examples of inverse monoids needed to understand this paper are the following. The *symmetric inverse monoids* $I(X)$ are the monoids of all partial bijections of the set X . When X has n elements, we denote the corresponding symmetric inverse monoid by I_n . We call the elements of X *letters*. Symmetric inverse monoids are Boolean inverse \wedge -monoids. We define an inverse monoid to be *semisimple* if it is isomorphic to a finite direct product of finite symmetric inverse monoids. They will play an important rôle in this paper. We also mention here two properties that arise naturally in our work. An inverse semigroup is said to be *fundamental* if the only elements that commute with every idempotent are themselves idempotents. An inverse monoid is said to be *factorizable* if every element is beneath an element in the group of units. Symmetric inverse monoids are fundamental and the finite ones are also factorizable. Thus semisimple inverse monoids are factorizable and fundamental. Our use of the word ‘semisimple’ was motivated by the theory of C^* -algebras and the following theorem. See [44] for a proof.

Theorem 2.1. *The finite fundamental Boolean inverse \wedge -monoids are precisely the semisimple inverse monoids.*

This paper is based on an exact analogy between semisimple inverse monoids and finite dimensional C^ -algebras.*

We conclude this section with some useful properties of meets and joins in inverse semigroups. The proofs of (1) and (2) may be found in [38], whereas (3) is folklore.

Lemma 2.2.

- (1) *We have that $s \sim t$ if and only if $s \wedge t$ exists and $\mathbf{d}(s \wedge t) = \mathbf{d}(s) \wedge \mathbf{d}(t)$ and $\mathbf{r}(s \wedge t) = \mathbf{r}(s) \wedge \mathbf{r}(t)$*
- (2) *In a distributive inverse monoid, if $a \vee b$ exists we have that*

$$\mathbf{d}(a \vee b) = \mathbf{d}(a) \vee \mathbf{d}(b) \text{ and } \mathbf{r}(a \vee b) = \mathbf{r}(a) \vee \mathbf{r}(b).$$
- (3) *In a distributive inverse monoid, if $a \vee b$ and $c \wedge (a \vee b)$ both exist then $c \wedge a$ and $c \wedge b$ both exist, the join $(c \wedge a) \vee (c \wedge b)$ exists and*

$$c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b).$$

2.2. Partial refinement monoids. Terminology in the area of partial algebras is not as well established as that of classical algebra. Moreover, the two areas of dimension theory and effect (and MV) algebras have often developed their own terminology for similar structures. We have opted to use mainly the terminology of dimension theory [27, 63] augmented by terminology from the theories of effect and MV-algebras to be found in [8, 18, 29, 30, 50, 51, 52, 53]. See also [28] for a modern categorical treatment of effect algebras.

Let E be a set equipped with a partially defined operation denoted \oplus together with a constant 0. If $a \oplus b$ is defined we write $\exists a \oplus b$. We define a *partial commutative monoid* to be such a set satisfying the following three axioms.

- (E1) $a \oplus b$ is defined if, and only if, $b \oplus a$ is defined and then they are equal.
- (E2) $(a \oplus b) \oplus c$ is defined if, and only if, $a \oplus (b \oplus c)$ is defined and then they are equal.
- (E3) For all $a \in E$, $\exists a \oplus 0$ and $a \oplus 0 = a$.

A partial commutative monoid that satisfies, in addition, the following axiom

- (E4) The *refinement property*¹:
if $a_1 \oplus a_2 = b_1 \oplus b_2$ then there exist elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that
 $a_1 = c_{11} \oplus c_{12}$ and $a_2 = c_{21} \oplus c_{22}$, and $b_1 = c_{11} \oplus c_{21}$ and $b_2 = c_{12} \oplus c_{22}$.

is called a *partial refinement monoid*. A partial commutative monoid that satisfies, in addition, the following axiom

- (E5) If $a \oplus b = 0$ then $a = 0$ and $b = 0$.

is said to be *conical* (or *positive*). A partial commutative monoid that satisfies, in addition, the following axiom

- (E6) If $a \oplus b = a \oplus c$ then $b = c$.

is said to be *cancellative*. To state the following two axioms, we need an additional constant denoted by 1.

- (E7) $a \oplus 1$ is defined if, and only if, $a = 0$.
- (E8) For each $a \in E$, there exists a unique $a' \in E$ such that $a \oplus a'$ exists and equals 1.

An *effect algebra* $(E, \oplus, 0, 1)$ is a structure satisfying (E1), (E2), (E7) and (E8). For the following see, for example, [25].

Lemma 2.3. *In an effect algebra, the axioms (E3), (E5), and (E6) hold automatically.*

Define $a \leq b$ if, and only if, $b = a \oplus c$ for some c . In an effect algebra $(E, \oplus, 0, 1)$, this relation is a partial order. A lattice-ordered effect algebra that also satisfies axiom (E4), the refinement property, is called an *MV-algebra* [25, 24]. In an MV-algebra, there is an everywhere defined binary operation

$$a \boxplus b = a \oplus (a' \wedge b).$$

It is possible to axiomatize MV-algebras in terms of this operation [24]. MV-algebras arose in the algebraic foundations of many-valued logics ([15, 52]).

2.3. Co-ordinatization. In this section, we shall define precisely what we mean by co-ordinatizing an MV-algebra by an inverse monoid. The idea behind our construction can be found sketched on page 131 of [60]. It is also related to the notion of coordinatizing a continuous geometry in the sense of von Neumann [54, 55].

An inverse monoid is said to be *completely semisimple*² if $e \mathcal{D} f \leq e$ implies $e = f$ for any idempotents e and f . In completely semisimple inverse monoids, we have that $\mathcal{D} = \mathcal{J}$. If $e \in E(S)$, a Boolean algebra, we denote by \bar{e} the complement of e . We say that \mathcal{D} *preserves complementation* if $e \mathcal{D} f$ implies that $\bar{e} \mathcal{D} \bar{f}$.

Lemma 2.4. *A Boolean inverse monoid in which \mathcal{D} preserves complementation is factorizable.*

Proof. Let $a \in S$. Put $e = a^{-1}a$ and $f = aa^{-1}$. Then $e \mathcal{D} f$. By assumption, $\bar{e} \mathcal{D} \bar{f}$. Thus there is an element b such that $\bar{e} \xrightarrow{b} \bar{f}$. The elements a and b are orthogonal and so have a join $g = a \vee b$. But $g^{-1}g = 1 = gg^{-1}$ and so g is an invertible element and, by construction, $a \leq g$. Thus S is factorizable. \square

¹Also called the Riesz Decomposition Property (RDP). See [58] and [18].

²There is no connection with the term ‘semisimple’ we introduced earlier.

Lemma 2.5. *A Boolean inverse monoid in which \mathcal{D} preserves complementation and in which $e \mathcal{D} 1$ implies that $e = 1$ is completely semisimple.*

Proof. Suppose that $e \mathcal{D} f \leq e$. Let $e \xrightarrow{a} f$. Observe that afa has domain e . Let $\bar{e} \xrightarrow{b} \bar{f}$. The elements afa and b are orthogonal. We may therefore form the orthogonal join $b \vee afa$. Its domain is $e \vee \bar{e} = 1$. By assumption, its range must also be the identity. Therefore $\bar{f} \vee afa^{-1} = 1$. This is an orthogonal join in a Boolean algebra and so $f = afa^{-1}$. That is, $(af)(af)^{-1} = f$. But $af \leq a$ and a has range f . We deduce that $a = af$. Thus $a^{-1}a = a^{-1}af$ and so $e = ef$. But $f = ef$ and so $e = f$, as required. \square

Let S be an arbitrary Boolean inverse monoid. Put $\mathbf{E}(S) = E(S)/\mathcal{D}$. We denote the \mathcal{D} -class containing the idempotent e by $[e]$. Define $[e] \oplus [f]$ as follows. Suppose that we can find idempotents $e' \in [e]$ and $f' \in [f]$ such that e' and f' are orthogonal. Then define $[e] \oplus [f] = [e' \vee f']$. Otherwise, the operation \oplus is undefined.

Proposition 2.6. *Let S be a Boolean inverse monoid.*

- (1) *The operation \oplus defined on $\mathbf{E}(S)$ is well-defined.*
- (2) *$(\mathbf{E}(S), \oplus, [0], [1])$ is a conical partial refinement monoid satisfying (E7).*
- (3) *$[e] \leq [f]$ if, and only if, $e \mathcal{D} i \leq f$ for some idempotent i .*
- (4) *If the partial algebra satisfies (E6), cancellativity, then the inverse monoid is completely semisimple.*
- (5) *If \mathcal{D} preserves complementation and $e \mathcal{D} 1$ implies that $e = 1$, then the partial algebra is an effect algebra.*
- (6) *The construction $S \mapsto \mathbf{E}(S)$ is functorial.*

Proof. (1) Let $e' \mathcal{D} e''$ and $f' \mathcal{D} f''$ where e' is orthogonal to f' , and e'' is orthogonal to f'' . We prove that $e' \vee f' \mathcal{D} e'' \vee f''$. By assumption, there are elements $e' \xrightarrow{a} e''$ and $f' \xrightarrow{b} f''$. The elements a and b are orthogonal and so $a \vee b$ exists. But $e' \vee f' \xrightarrow{a \vee b} e'' \vee f''$.

(2) It is immediate that (E1) holds.

To prove axiom (E2), ironically, takes a bit of work. Suppose that $\exists([e] \oplus [f]) \oplus [g]$. Then $\exists[e] \oplus [f]$ and so we may find $e \xrightarrow{a} e'$ and $f \xrightarrow{b} f'$ such that e' and f' are orthogonal. By definition, $[e] \oplus [f] = [e' \vee f']$. Since $\exists[e' \vee f'] \oplus [g]$, we may find $e' \vee f' \xrightarrow{c} i$ and $g \xrightarrow{d} g'$ such that i and g' are orthogonal. It follows that

$$([e] \oplus [f]) \oplus [g] = [i \vee g'].$$

Define $x = ce'$ and $y = cf'$. Then

$$e' \xrightarrow{x} \mathbf{r}(x) \text{ and } f' \xrightarrow{y} \mathbf{r}(y).$$

Since i is orthogonal to g' and $\mathbf{r}(y) \leq i$, we have that $\mathbf{r}(y)$ and g' are orthogonal. In addition, yb has domain f and range $\mathbf{r}(y)$. It follows that $\exists[f] \oplus [g]$ and it is equal to $[\mathbf{r}(y) \vee g']$. Observe next that $\mathbf{r}(x)$ is orthogonal to $\mathbf{r}(y)$ and, since $\mathbf{r}(x) \leq i$ it is also orthogonal to g' . It follows that $\mathbf{r}(x)$ is orthogonal to $\mathbf{r}(y) \vee g'$. But xa has domain e and range $\mathbf{r}(x)$. It follows that $\exists[e] \oplus [\mathbf{r}(y) \vee g']$ is defined and equals $[\mathbf{r}(x) \vee \mathbf{r}(y) \vee g']$. But $\mathbf{r}(x) \vee \mathbf{r}(y) = i$. It follows that we have shown

$$\exists[e] \oplus ([f] \oplus [g])$$

and that it equals $([e] \oplus [f]) \oplus [g]$.

The reverse implication follows by symmetry.

It is immediate that (E3) holds.

We prove that (E4) holds. Let $[e_1] \oplus [e_2] = [f_1] \oplus [f_2]$ where we assume, without loss of generality, that e_1 is orthogonal to e_2 , and f_1 is orthogonal to f_2 . Let

$e_1 \vee e_2 \xrightarrow{x} f_1 \vee f_2$. Clearly

$$x = (f_1 \vee f_2)x(e_1 \vee e_2).$$

Put

$$x_1 = f_1 x e_1, \quad x_2 = f_1 x e_2, \quad x_3 = f_2 x e_1, \quad x_4 = f_2 x e_2.$$

Then

$$x = x_1 \vee x_2 \vee x_3 \vee x_4,$$

an orthogonal join. Define also

$$a_{11} = [\mathbf{d}(x_1)], \quad a_{12} = [\mathbf{d}(x_2)], \quad a_{21} = [\mathbf{d}(x_3)], \quad a_{22} = [\mathbf{d}(x_4)].$$

Observe that $\mathbf{d}(x_1), \mathbf{d}(x_3) \leq e_1$. Thus $\mathbf{d}(x_1) \vee \mathbf{d}(x_3) = e_1$ and $\mathbf{d}(x_2) \vee \mathbf{d}(x_4) = e_2$. Thus $a_{11} \oplus a_{21} = [e_1]$ and $a_{12} \oplus a_{22} = [e_2]$. Similarly, $f_1 = \mathbf{r}(x_1) \vee \mathbf{r}(x_2)$ and $f_2 = \mathbf{r}(x_3) \vee \mathbf{r}(x_4)$. Thus $[f_1] = a_{11} \oplus a_{12}$ and $[f_2] = a_{21} \oplus a_{22}$.

(E5) holds because if the join of two idempotents is 0 then both idempotents must be 0, and the only idempotent \mathcal{D} -related to 0 is 0 itself.

(E7) holds because the only idempotent orthogonal to the identity is 0, and the only idempotent \mathcal{D} -related to 0 is 0 itself.

(3) Suppose that $e \xrightarrow{x} i \leq f$. We may find an idempotent j such that $f = i \vee j$ and $i \wedge j = 0$. Then $[e] \oplus [j] = [f]$ and so $[e] \leq [f]$. Conversely, suppose that $[e] \leq [f]$ where e and f are idempotents. Then there exists an idempotent g such that $[e] \oplus [g] = [f]$. By definition, there are elements $e \xrightarrow{a} e'$ and $g \xrightarrow{b} g'$ such that $e' \vee f' \mathcal{D} f$. But then $e \mathcal{D} e' \leq f$, as required.

(4) Suppose that $e \mathcal{D} f \leq e$. Then $e = f \vee e \setminus f$. Put $g = e \setminus f$. Thus $[e] = [e] \oplus [g]$. But then $[e] \oplus [0] = [e] \oplus [g]$. By cancellation, we get that $[0] = [g]$ and so $g = 0$ from which we deduce that $e = f$, as required.

(5) Define $[e]' = [\bar{e}]$. This is well-defined since \mathcal{D} preserves complementation. Clearly, $[e] \oplus [\bar{e}] = [1]$. Suppose that $[e] \oplus [f] = [1]$. By definition, we have idempotents i and j such that $i \mathcal{D} e$ and $j \mathcal{D} f$ and $i \vee j \mathcal{D} 1$. By assumption, $i \vee j = 1$, an orthogonal join. It follows that $j = \bar{i}$. But $i \mathcal{D} e$ implies that $\bar{i} \mathcal{D} \bar{e}$. Thus $[f] = [e]'$, as required.

(6) Let $\theta: S \rightarrow T$ be a morphism of Boolean inverse monoids. Any morphism preserves the \mathcal{D} -relation and so we may define $\theta^*: \mathbf{E}(S) \rightarrow \mathcal{E}(T)$ by $\theta^*([e]) = [\theta(e)]$. Suppose that $[e] \oplus [f]$ is defined. Then there exist idempotents e' and f' such that $e \mathcal{D} e'$ and $f \mathcal{D} f'$ and where e' and f' are orthogonal. Thus $[e] \oplus [f] = [e' \vee f']$. But orthogonality is preserved by morphisms and $\theta(e' \vee f') = \theta(e') \vee \theta(f')$. It follows that $\theta^*([e] \oplus [f]) = \theta^*([e]) \oplus \theta^*([f])$. Morphisms are also morphisms of Boolean algebras and so $\theta^*([e]') = \theta^*([e])'$. It is now straightforward to check that we have actually defined a functor from Boolean inverse monoids to partial algebras. \square

We shall call $\mathbf{E}(S)$ equipped with the partially defined binary operation \oplus the *partial algebra* associated with the inverse monoid S . By Lemma 2.5 and Proposition 2.6, we immediately deduce the following.

Corollary 2.7. *Let S be a Boolean inverse monoid that is completely semisimple and in which \mathcal{D} preserves complementation. Then the partial algebra $(\mathbf{E}(S), \oplus, [0], [1])$ is an effect algebra satisfying the refinement property.*

In the light of the above result, it is convenient to define a *Foulis monoid* to be a completely semisimple Boolean inverse monoid in which \mathcal{D} preserves complementation. The construction $S \mapsto \mathbf{E}(S)$ is in fact a functor from the category of Foulis monoids to the category of effect algebras with the refinement property. Since a Foulis monoid is completely semisimple, $\mathcal{D} = \mathcal{J}$. It follows that we may identify $\mathbf{E}(S)$ with S/\mathcal{J} , the poset of principal ideals of S . We say that an effect algebra E can be *co-ordinatized* if there is a Foulis monoid S such that E is isomorphic

to S/\mathcal{J} as an effect algebra. An inverse monoid with zero S is said to satisfy the *lattice condition* if S/\mathcal{J} is a lattice. If S is a Foulis monoid satisfying the lattice condition then S/\mathcal{J} is in fact an MV-algebra. Using the definitions we have made, the goal of this paper can now be precisely stated:

For each countable MV-algebra E , show that there is a Foulis monoid S satisfying the lattice condition such that E is isomorphic to S/\mathcal{J} as an MV-algebra.

2.4. Co-ordinatizing finite MV-algebras. In this section, we shall prove that all finite MV-algebras are co-ordinatizable, a result motivating everything we do subsequently. But we also take the opportunity to introduce some ideas that will also be useful to us later.

Let S be a Boolean inverse monoid. The following definition was suggested by [14]. An *invariant mean* for S is a function $\mu: E(S) \rightarrow [0, 1]$ such that the following axioms hold:

- (IM1): $\mu(1) = 1$.
- (IM2): For any $s \in S$, we have that $\mu(s^{-1}s) = \mu(ss^{-1})$.
- (IM3): If e and f are orthogonal idempotents we have that $\mu(e \vee f) = \mu(e) + \mu(f)$.

Observe that since 0 is orthogonal to itself $\mu(0) = 0$.

We shall say that an invariant mean is *good* if for all $e, f \in E(S)$ if $\mu(e) \leq \mu(f)$ then there exists e' such that $\mu(e) = \mu(e')$ and $e' \leq f$. This definition is adapted from [3]. Finally, we say that an invariant mean *reflects the \mathcal{D} -relation* if $\mu(e) = \mu(f)$ implies that $e \mathcal{D} f$.

Lemma 2.8. *Let S be a Boolean inverse monoid equipped with a good invariant mean μ that reflects the \mathcal{D} -relation. Then S is completely semisimple, \mathcal{D} preserves complementation, and S/\mathcal{J} is linearly ordered.*

Proof. Observe first that if e and f are any idempotents such that $e \leq f$ then $\mu(e) \leq \mu(f)$. If $e \neq f$, we may suppose that $e < f$. Then $f = e \vee (f\bar{e})$, an orthogonal join. Thus $\mu(f) = \mu(e) + \mu(f\bar{e})$. It follows that $\mu(f) \geq \mu(e)$. Next observe that $\mu(e) = 0$ implies that $e = 0$. We have that $\mu(e) = \mu(0)$ and so, by assumption, $e \mathcal{D} 0$. It is now immediate that $e = 0$.

We show that S is completely semisimple. Suppose that $e \mathcal{D} f \leq e$. Then $\mu(e) = \mu(f)$. Since $f \leq e$, we may write $e = f \vee (ef)$, an orthogonal join. Thus $\mu(ef) = 0$ and so, by the above, $ef = 0$. It follows that $e = f$, as required.

We show that \mathcal{D} preserves complementation. Suppose that $e \mathcal{D} f$. Then $\mu(e) = \mu(f)$. We have that $\mu(\bar{e}) = 1 - \mu(e)$ and $\mu(\bar{f}) = 1 - \mu(f)$. Thus $\mu(\bar{e}) = \mu(\bar{f})$. Hence, by assumption, $\bar{e} \mathcal{D} \bar{f}$, as required.

Finally, we show that S/\mathcal{J} is linearly ordered thus, in particular, S satisfies the lattice condition. Let e and f be arbitrary idempotents. Without loss of generality, we may assume that $\mu(e) \leq \mu(f)$. Since the invariant mean μ is good, there is an idempotent e' such that $\mu(e) = \mu(e')$ and $e' \leq f$. By assumption, $e \mathcal{D} e'$. It follows that $SeS \subseteq SfS$. \square

A natural example of a Boolean inverse monoid equipped with an invariant mean is the symmetric inverse monoid I_n . Define $\mu(1_A) = \frac{|A|}{n}$. In other words, we assign probability $\frac{1}{n}$ to each letter. The proof of the following can easily be deduced from the case of finite symmetric inverse monoids and the closure of the stated properties under finite direct products.

Lemma 2.9. *Semisimple inverse monoids are completely semisimple and \mathcal{D} preserves complementation.*

Our first main theorem is the following.

Theorem 2.10. *Every finite MV-algebra can be co-ordinatized by a semisimple inverse monoid.*

Proof. The core of the proof is the following. Let I_n be the finite symmetric inverse monoid on n letters. By Lemma 2.9, such monoids are Foulis monoids. It is well-known that they also satisfy the lattice condition, but we shall prove this explicitly since the method we use is important for what we do later. Define $\kappa: S/\mathcal{J} \rightarrow \mathbb{Z}$ by $\kappa[1_A] = |A|$. In other words, we map an idempotent to the cardinality of the set on which it is partial identity. Observe that two idempotents are \mathcal{D} -related if, and only if, these cardinalities are the same. It follows that κ is a bijection from I_n/\mathcal{J} to the set $\mathbf{n} = \{0, 1, \dots, n\}$. It is trivial that it induces an order isomorphism when the set \mathbf{n} carries the usual order. It follows that the lattice condition is satisfied with the lattice operations being \min and \max . The partial operation \oplus translates into partial addition: if $r, s \in \mathbf{n}$ then $r \oplus s = r + s$ if $r + s \leq n$, otherwise it is undefined. The prime operation translates into $s' = n - s$. We now describe the operation \boxplus . By definition

$$r \boxplus s = r + \min(r', s).$$

We consider two cases. Suppose first that $r + s \leq n$. Then $s \leq n - r = r'$. It follows that in this case $r \boxplus s = r + s$. Next suppose that $r + s > n$. Then $s > n - r = r'$. It follows that in this case $r \boxplus s = n$. We have therefore shown that I_n/\mathcal{J} gives rise to the MV-algebra known as the *Lukasiewicz chain* L_{n+1} [52].

To prove the full theorem, we now use the result that every finite MV-algebra is a finite direct product of Lukasiewicz chains. See [16, Proposition 3.6.5] or part 2 of [52, Theorem 11.2.4]. Such algebras can clearly be co-ordinatized by finite direct products of finite symmetric inverse monoids and so by semisimple inverse monoids. \square

3. AF INVERSE MONOIDS

In this section, we shall define the class of approximately finite (AF) inverse monoids and derive their basic properties. The term *AF inverse semigroup* was also used in [57] for inverse semigroups generated according to a quite complex recipe, whereas in [36], Kumjian defines *AF localizations* which he states may be viewed ‘in some sense’ as inductive limits of finite localizations. Our definition is simpler than either of the above definitions and shadows that of the definition of AF C^* -algebras. It works because we use the correct definition of morphism between semisimple inverse monoids. In any event, our AF monoids will turn out to be Foulis monoids, and they will provide one of the key ingredients in proving our main theorem. Good sources for Bratteli diagrams and the construction of C^* -algebras from them are [19, 26].

Fundamental to our work is the choice of the correct morphisms. A *morphism* between Boolean inverse \wedge -monoids is a unital homomorphism that maps zeros to zeros, preserves all compatible binary joins and preserves all binary meets. We define the *kernel* of a morphism to be all the elements that are mapped to zero.

Lemma 3.1. *A morphism $\theta: S \rightarrow T$ between Boolean inverse \wedge -monoids is injective if, and only if, its kernel is zero.*

Proof. Only one direction needs proving. Suppose that the kernel of θ is zero but there exist non-zero elements a and b such that $\theta(a) = \theta(b)$. Since $\theta(a \wedge b) = \theta(a) \wedge \theta(b)$, we have that $\theta(a \wedge b) = \theta(a) = \theta(b)$. Thus without loss of generality, we may as well assume that $b \leq a$. We may construct an element x such that $a = b \vee x$ and $b \wedge x = 0$; define $x = a(a^{-1}a(b^{-1}b))$. Then $\theta(a) = \theta(b) \vee \theta(x)$ and

$\theta(b) \wedge \theta(x) = 0$. Since $\theta(a) = \theta(b)$ we deduce that $\theta(x) \leq \theta(a)$. But then $\theta(x) = 0$ and so $x = 0$. We deduce that $a = b$, as required. \square

3.1. The construction of AF monoids. In the symmetric inverse monoid I_n , we denote by e_{ij} the partial bijection with domain $\{j\}$ and codomain $\{i\}$. The elements e_{ii} are idempotents. Every element of I_n can be written as a unique orthogonal join of the elements e_{ij} . In the case of idempotents, only the elements of the form e_{ii} are needed. Consider now the set of all $n \times n$ matrices whose entries are drawn from $\{0, 1\}$ in which each row and each column contains at most one non-zero element. The set of all such matrices is denoted by R_n and called the set of *rook matrices* [61]. Fix an ordering of the set of letters of an n -element set. For each $f \in I_n$ define $M(f)_{ij} = 1$ if $i = f(j)$ and 0 otherwise. In this way, we obtain a bijection between I_n and R_n which maps the identity function to the identity matrix and which is a homomorphism between function composition and matrix multiplication. Thus the rook matrices R_n provide isomorphic copies of I_n . We have that $f \leq g$, the natural partial order, if and only if $M(f)_{ij} = 1 \Rightarrow M(g)_{ij} = 1$. The meet $f \wedge g$ corresponds to the *freshman product*³ of $M(f)$ and $M(g)$. The elements e_{ij} correspond to those rook matrices which are matrix units. Let A and B be rook matrices of sizes $m \times m$ and $n \times n$, respectively. We denote by $A \oplus B$ the $(m+n) \times (m+n)$ rook matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

We may iterate this construction. We write $sA = A \oplus \dots \oplus A$ where the sum has s summands. There is no ambiguity with scalar multiplication because such multiplication is not defined for rook matrices. More generally, we can form sums such as $s_1 A_1 \oplus \dots \oplus s_m A_m$. There are many isomorphisms between I_n and R_n but the only ones that we will need are those determined by choosing a total ordering of the letters $\{1, \dots, n\}$. We shall call such isomorphisms *letter isomorphisms*. We shall also be interested in isomorphisms from $I_{n_1} \times \dots \times I_{n_k}$ to $R_{n_1} \times \dots \times R_{n_k}$ induced by letter isomorphisms from I_{n_i} to R_{n_i} . We shall also refer to these as letter isomorphisms.

Our first goal is to classify morphisms between semisimple inverse monoids. We begin with a lemma that is a rare example of an arithmetic result in semigroup theory.

Lemma 3.2. *There is a morphism from I_m to I_n if, and only if, $m \mid n$.*

Proof. Assume first that there is a morphism $\theta: I_m \rightarrow I_n$. We may write $1 = \bigvee_{i=1}^m e_{ii}$ an orthogonal join. Since θ is a morphism, we have that $\theta(1) = 1$ and $\theta(\bigvee_{i=1}^m e_{ii}) = \bigvee_{i=1}^m \theta(e_{ii})$. Thus $1 = \bigvee_{i=1}^m \theta(e_{ii})$. Orthogonality is preserved by homomorphisms that map zeros to zeros. Thus the union on the righthandside above is an orthogonal union. Clearly $e_{ii} \mathcal{D} e_{jj}$ for all i and j . Thus $\theta(e_{ii}) \mathcal{D} \theta(e_{jj})$. But two idempotents in a symmetric inverse monoid are \mathcal{D} -related precisely when their domains of definition have the same cardinality. Thus $\theta(e_{ii}) = 1_{A_i}$ where the sets A_1, \dots, A_m are pairwise disjoint and have the same cardinality s , say. It follows that $n = sm$, and so $m \mid n$, as claimed.

To prove the converse, suppose that $n = sm$. Choose letter isomorphisms from I_m to R_m and I_n to R_n . Define a map from R_m to R_n as follows $A \mapsto sA$. It is easy to check that this is a morphism. Thus we get a morphism from I_m to I_n , as claimed. \square

If $n = sm$, then the morphism from R_m to R_n defined by $A \mapsto sA$ is called a *standard morphism*. Our next result says that, up to letter isomorphisms, all such morphisms are described by standard morphisms.

³That is, corresponding entries are multiplied.

Lemma 3.3. *Suppose that $m \mid n$ where $n = sm$. Let $\theta: I_m \rightarrow I_n$ be a morphism and let $\alpha: I_m \rightarrow R_m$ be a letter isomorphism. Then there is a standard map $\sigma: R_m \rightarrow R_n$ and a letter isomorphism $\beta: I_n \rightarrow R_n$ such that $\theta = \beta^{-1}\sigma\alpha$. In particular, every morphism from I_m to I_n is isomorphic to a standard map.*

Proof. Let $\theta: I_m \rightarrow I_n$ be a morphism. We begin as in the proof of Lemma 3.2. Choose any ordering of the letters of I_m and let $\alpha: I_m \rightarrow R_m$ be the corresponding isomorphism. We may suppose that the letters are labelled $1, \dots, m$. Define the elements e_{ij} relative to that ordering. Let $1 = \bigvee_{i=1}^m e_{ii}$. Then $1 = \bigvee_{i=1}^m \theta(e_{ii})$ where $\theta(e_{ii}) = 1_{A_i}$ and the sets A_1, \dots, A_m are pairwise disjoint and have the same cardinality s . Let $A_i = \{x_{i1}, \dots, x_{is}\}$ where $i = 1, \dots, m$. Now order the elements of $\bigcup_{i=1}^m A_i$ as follows

$$x_{11}, x_{21}, \dots, x_{m1}, \dots, x_{1s}, \dots, x_{ms}.$$

With this ordering, construct an isomorphism $\beta: I_n \rightarrow R_n$. Let $\sigma: I_m \rightarrow I_n$ be the standard map $A \mapsto sA$. We claim that $\theta = \beta^{-1}\sigma\alpha$. It's enough to verify this for the partial bijections e_{ij} . We have that

$$e_{jj} \xrightarrow{e_{ij}} e_{ii}.$$

Thus $\theta(e_{ij})$ has domain the domain of definition of $\theta(e_{jj})$ and image the image of definition of $\theta(e_{ii})$. The domain of definition of $\theta(e_{jj})$ is the set A_j . If the rook matrix of e_{jj} is the matrix M which has one non-zero entry in row j and column j , the matrix of $\theta(e_{jj})$ relative to the above ordering of letters will be sM . The proof now readily follows. \square

If S is an inverse monoid and e is any idempotent then eSe is an inverse subsemigroup called a *local submonoid* (sic). Consider now the symmetric inverse monoid I_n . Then an idempotent $e = 1_A$ where $A \subseteq \{1, \dots, n\}$. It follows that the local submonoid $eI_n e$ is simply I_A , the symmetric inverse monoid on the set of letters A .

Let $s_1m(1) + \dots + s_km(k) = n$, where the s_i are non-negative integers. Define the corresponding *standard morphism* $\sigma: R_{m(1)} \times \dots \times R_{m(k)} \rightarrow R_n$ by $\sigma((A_1, \dots, A_k)) = s_1A_1 \oplus \dots \oplus s_kA_k$. We may now classify morphisms from semisimple inverse monoids to symmetric inverse monoids.

Lemma 3.4.

- (1) *There is a morphism from $I_{m(1)} \times \dots \times I_{m(k)}$ to I_n if, and only if, there exist non-negative integers s_1, \dots, s_k such that $s_1m(1) + \dots + s_km(k) = n$.*
- (2) *For each morphism $\theta: I_{m(1)} \times \dots \times I_{m(k)} \rightarrow I_n$ and for each letter isomorphism $\alpha: I_{m(1)} \times \dots \times I_{m(k)} \rightarrow R_{m(1)} \times \dots \times R_{m(k)}$ there exists a letter isomorphism $\beta: I_n \rightarrow R_n$ and a standard morphism $\sigma: R_{m(1)} \times \dots \times R_{m(k)} \rightarrow R_n$ such that $\theta = \beta^{-1}\sigma\alpha$.*

Proof. (1) Denote the set of letters of I_n by X . Put $S = I_{m(1)} \times \dots \times I_{m(k)}$. The identity of this monoid is the k -tuple of identities whose i th component is the identity of $I_{m(i)}$. Define \mathbf{e}_i to be the idempotent of S all of whose elements are zero except the i th which is the identity of $I_{m(i)}$. Then $1 = \bigvee_{i=1}^k \mathbf{e}_i$ is an orthogonal join. Thus $1 = \bigvee_{i=1}^k \theta(\mathbf{e}_i)$ is an orthogonal join and the identity function on X . Let $\theta(\mathbf{e}_i) = 1_{X_i}$. Denote the cardinality of X_i by a_i . The non-empty X_i form a partition of X . It follows that $n = a_1 + \dots + a_k$. For each i , where $X_i \neq \emptyset$, we have that $\theta(\mathbf{e}_i)I_n\theta(\mathbf{e}_i) = I_{X_i}$, a symmetric inverse monoid on a_i letters. Now the morphism θ restricts to a morphism θ_i from $\mathbf{e}_iS\mathbf{e}_i$ to $\theta(\mathbf{e}_i)I_n\theta(\mathbf{e}_i) = I_{X_i}$. But we have that $\mathbf{e}_iS\mathbf{e}_i \cong I_{m(i)}$. We therefore have an induced morphism from $I_{m(i)}$ to I_{a_i} . Thus by Lemma 3.2, $a_i = s_i m(i)$ for some non-zero s_i . Hence $s_1m(1) + \dots + s_km(k) = n$. The converse is proved using a standard morphism defined as above.

(2) We continue with the notation introduced in part (1). Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a letter isomorphism from $I_{m(1)} \times \dots \times I_{m(k)}$ to $R_{m(1)} \times \dots \times R_{m(k)}$. In what follows, we need only deal with the i where $X_i \neq \emptyset$. Let $\iota_i: I_{m(i)} \rightarrow S$ be the obvious embedding. Put $\theta_i = \theta_{\iota_i}$. Then $\theta_i: I_{m(i)} \rightarrow I_{X_i}$. There is therefore a letter isomorphism $\beta_i: I_{X_i} \rightarrow R_{a_i}$ obtained through a specific ordering of the elements of X_i and the standard map $\sigma_i: R_{m(i)} \rightarrow R_{a_i}$ given by $A \mapsto s_i A$ such that $\theta_i = \beta_i^{-1} \sigma_i \alpha_i$. We order the letters of I_n as X_1, \dots, X_k with the ordering within each X_i chosen as above. Define $\beta: I_n \rightarrow R_n$ to be the corresponding letter isomorphism. Then $\sigma = \sigma_1 \oplus \dots \oplus \sigma_k$. \square

Remark 3.5. Observe that

$$s_i = \frac{|\theta(\mathbf{e}_i)|}{m(i)}$$

where $|\theta(\mathbf{e}_i)|$ denotes the cardinality of the set on which the idempotent $\theta(\mathbf{e}_i)$ is defined.

We suppose we are given $I_{m(1)} \times \dots \times I_{m(k)}$ and $I_{n(1)} \times \dots \times I_{n(l)}$. Put $\mathbf{m} = (m(1) \dots m(k))^T$ and $\mathbf{n} = (n(1) \dots n(l))^T$. Assume that we are given an $l \times k$ matrix M , where $M_{ij} = s_{ij}$, non-negative natural numbers, such that $M\mathbf{m} = \mathbf{n}$. Then we define a standard map σ from $R_{m(1)} \times \dots \times R_{m(k)}$ to $R_{n(1)} \times \dots \times R_{n(l)}$ by

$$\begin{pmatrix} A_1 \\ \dots \\ A_k \end{pmatrix} \mapsto M \begin{pmatrix} A_1 \\ \dots \\ A_k \end{pmatrix}$$

Proposition 3.6. *Given a morphism $\theta: S = I_{m(1)} \times \dots \times I_{m(k)} \rightarrow I_{n(1)} \times \dots \times I_{n(l)} = T$ and a letter isomorphism $\alpha: I_{m(1)} \times \dots \times I_{m(k)} \rightarrow R_{m(1)} \times \dots \times R_{m(k)}$ there is a letter isomorphism $\beta: I_{n(1)} \times \dots \times I_{n(l)} \rightarrow R_{n(1)} \times \dots \times R_{n(l)}$ and a standard map $\sigma: R_{m(1)} \times \dots \times R_{m(k)} \rightarrow R_{n(1)} \times \dots \times R_{n(l)}$ such that $\theta = \beta^{-1} \sigma \alpha$.*

Proof. We use the l projection morphisms from T to each of $I_{n(1)}, \dots, I_{n(l)}$ composed with θ to get morphisms from S to each of $I_{n(1)}, \dots, I_{n(l)}$ in turn. We now apply Lemma 3.4. The separate results can now easily be combined to prove the claim. \square

The data involved in describing a morphism from $I_{m(1)} \times \dots \times I_{m(k)}$ to $I_{n(1)} \times \dots \times I_{n(l)}$ can be encoded by means of a directed graph which we shall call a *diagram*. We draw k vertices, labelled $m(1) \dots m(k)$, in a line, the *upper vertices*, and then we draw l vertices, labelled $n(1) \dots n(l)$, on the line below, the *lower vertices*. We join the vertex labelled $m(j)$ to the vertex labelled $n(i)$ by means of s_{ij} directed edges. We require such graphs to satisfy the arithmetic conditions $n(i) = s_{i1}m(1) + \dots + s_{ik}m(k)$. We call these the *combinatorial conditions*. In other words, the matrix M defined above is the adjacency matrix where the upper vertices label the columns and the lower vertices label the rows.

Remark 3.7. In a diagram, each lower vertex is the target of at least one edge. This is immediate by Lemma 3.4.

Lemma 3.8. *Let $\sigma: S = I_{m(1)} \times \dots \times I_{m(k)} \rightarrow I_{n(1)} \times \dots \times I_{n(l)} = T$ be a standard map. Then σ is injective if, and only if, every upper vertex is the source of some directed edge.*

Proof. Without loss of generality, suppose that the upper vertex $m(1)$ is not the source of any edge. Then all the elements $I_{m(1)} \times \{0\} \times \dots \times \{0\}$ are in the kernel of σ and so, in particular, σ is not injective. Now suppose that every upper vertex is the source of some edge. Then clearly σ has kernel equal to zero. We now use Lemma 3.1 to deduce that σ is injective. \square

We now recall a standard definition [9]. A *Bratteli diagram* is an infinite directed graph $B = (V, E)$ with vertex-set V and edge-set E such that $V = \bigcup_{i=0}^{\infty} V(i)$ and $E = \bigcup_{i=1}^{\infty} E(i)$ are partitions of the respective sets into finite blocks, in the case of the vertices called *levels*, such that

- (1) $V(0)$ consists of one vertex v_0 we call the *root*.
- (2) Edges are only defined from $V(i)$ to $V(i+1)$, that is *adjacent levels*, and there are only finitely many edges from one level to the next.
- (3) Each vertex is the source of an edge and each vertex, apart from the root, is the target of an edge.

Remark 3.9. We have proved that each injective morphism between two semisimple inverse monoids determines a diagram that satisfies the condition to be adjacent levels in a Bratteli diagram.

Let B be a Bratteli diagram. For each vertex v we define its *size* s_v to be the number of directed paths from the root v_0 in B to v . We now associate a semisimple inverse monoid with each level of the Bratteli diagram. With the root vertex, we associate $S_0 = I_1$, the two-element Boolean inverse \wedge -monoid. With level $i \geq 2$, we associate the inverse monoid S_i . This is constructed as follows. List the k vertices of level i and then their respective sizes as $m(1), \dots, m(k)$. We put $S_i = I_{m(1)} \times \dots \times I_{m(k)}$. We now show how to define a morphism from S_i to S_{i+1} . List the l vertices of level $i+1$ and then their respective weights as $n(1), \dots, n(l)$. In the Bratteli diagram, the vertex $m(j)$ will be joined to the vertex $n(i)$ by s_{ij} edges. The following is proved using a simple counting argument.

Lemma 3.10. *Adjacent levels of a Bratteli diagram satisfy the combinatorial conditions.*

It follows that we may define a standard morphism σ_i from S_i to S_{i+1} . This will be injective by Lemma 3.8. We have therefore constructed a sequence of injective morphisms between semisimple inverse monoids

$$S_0 \xrightarrow{\sigma_0} S_1 \xrightarrow{\sigma_1} S_2 \xrightarrow{\sigma_2} \dots$$

We shall now describe direct limits of Boolean inverse monoids. We begin with a well-known construction in semigroup theory. Let

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \dots$$

be a sequence of inverse monoids and injective morphisms. We use the dual order on \mathbb{N} . If $i, j \in \mathbb{N}$ denote by $i \wedge j$ the maximum element in of $\{i, j\}$. For $j < i$ define $\tau_j^i = \tau_{j-1} \dots \tau_i$. Thus $\tau_{i+1}^i = \tau_i$. Define τ_i^i to be the identity function on S_i . Clearly, if $k \leq j \leq i$ then $\tau_k^i = \tau_k^j \tau_j^i$. Put $S = \bigsqcup_{i=0}^{\infty} S_i$, a disjoint union of sets. Let $a, b \in S$ where $a \in S_i$ and $b \in S_j$. Define

$$a \cdot b = \tau_{i \wedge j}^i(a) \tau_{i \wedge j}^j(b).$$

Then (S, \cdot) is a semigroup. We shall usually represent multiplication by concatenation. Observe that the set of idempotents of S is the union of the set of idempotents of each of the S_i . It is routine that idempotents commute. In addition, S is regular. It follows that S is an inverse semigroup. The inverse of $a \in S$ where $a \in S_i$ is simply its inverse in S_i . The identity element of S_0 is the identity for the semigroup S . The monoid S is said to be an ω -chain of inverse monoids.

Remark 3.11. The semigroup S does not have a zero. Instead, the set of zeros from each S_i forms an ideal \mathcal{Z} in S . If we form the quotient monoid, S/\mathcal{Z} then essentially all the elements of $S \setminus \mathcal{Z}$ remain the same whereas the elements of \mathcal{Z} are rolled up into one zero.

Denote the identity of S_i by e_i . Put $\mathcal{E} = \{e_i : i \in \mathbb{N}\}$. Then \mathcal{E} forms a subsemigroup of the semigroup S and is a subset of the centralizer of S . For each $a \in S$, there exists $e \in \mathcal{E}$ such that $a = ea = ae$. Define $a \equiv b$ if, and only if, $ae = be$ for some $e \in \mathcal{E}$. Then \equiv is a congruence on S and the quotient is an inverse monoid with zero.

Lemma 3.12. *Let*

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \dots$$

be a sequence of Boolean inverse \wedge -monoids and injective morphisms. Then the direct limit $\varinjlim S_i$ is a Boolean inverse \wedge -monoid. In addition, we have the following.

- (1) *If all the S_i are fundamental then $\varinjlim S_i$ is fundamental.*
- (2) *If all the S_i are factorizable then $\varinjlim S_i$ is factorizable.*
- (3) *If all the S_i are completely semisimple then $\varinjlim S_i$ is completely semisimple.*
- (4) *If all the S_i have the property that \mathcal{D} preserves complementation then $\varinjlim S_i$ satisfies the property that \mathcal{D} preserves complementation.*
- (5) *The group of units of $\varinjlim S_i$ is the direct limit of the groups of units of the S_i .*

Proof. We construct ω -chain of inverse monoids S , as above. Let $j \leq i$ and let $b \in S_j$ and $a \in S_i$. Then $b = \tau_j^i(a)$ if, and only if, $b = a \cdot e_j$. It follows, in particular, that $b \leq a$. Let $a \in S_i$ and $b \in S_j$. Then there is $l \leq i, j$ such that $\tau_l^i(a) = \tau_l^j(b)$ if, and only if, $e_l a = e_l b$. Define $a \equiv b$ if, and only if, there exists $e \in \mathcal{E}$ such that $ea = eb$. Then, as above, \equiv is a congruence on the inverse semigroup S . It is idempotent-pure because the τ_i are injective. We denote the \equiv -class containing the element a by $[a]$. We denote the set of \equiv -classes by S_∞ . All the elements in \mathcal{Z} are identified and so S_∞ is an inverse monoid with zero. Observe that the product is given by

$$[a][b] = [\tau_{i \wedge j}^i(a) \tau_{i \wedge j}^j(b)].$$

Let $[a], [b] \in S_\infty$ where $a \in S_i$ and $b \in S_j$. Then $[a] \sim [b]$ if and only if $\tau_{i \wedge j}^i(a) \sim \tau_{i \wedge j}^j(b)$. It is now routine to check that S_∞ has binary compatible joins, and that multiplication distributes over such joins. Let $[a], [b] \in S_\infty$ where $a \in S_i$ and $b \in S_j$. Put $c = \tau_{i \wedge j}^i(a) \wedge \tau_{i \wedge j}^j(b)$. We show that $[c] = [a] \wedge [b]$. Observe that if $x, y \in S_l$ and $x \leq y$ then $[x] \leq [y]$. We have that $[a] = [\tau_{i \wedge j}^i(a)]$ and $[b] = [\tau_{i \wedge j}^j(b)]$. Clearly $[c] \leq [a], [b]$. It is now routine to check that if $[d] \leq [a], [b]$ then $[d] \leq [c]$. If $[e]$ is an idempotent then the operation $\overline{[e]} = [\overline{e}]$ is well-defined and $[e] \wedge \overline{[e]} = [0]$ and $[e] \vee \overline{[e]} = [1]$. We have therefore shown that S_∞ is a Boolean inverse \wedge -monoid.

Define $\phi_i : S_i \rightarrow S_\infty$ by $s \mapsto [s]$. This map is evidently a morphism and whenever $j \leq i$ we have that $\phi_j \tau_j^i = \phi_i$. Now let T be a Boolean inverse \wedge -monoid such that there are morphisms $\theta_i : S_i \rightarrow T$ such that whenever $j \leq i$ we have that $\theta_j \tau_j^i = \theta_i$. Define $\psi : S_\infty \rightarrow T$ by $\psi([a]) = \theta_i(a)$ if $a \in S_i$. That this is a well-defined morphism witnessing that S_∞ is indeed the direct limit is now routine.

(1) The proof of this is straightforward. In particular, it uses the fact that if the image of an element under an injective morphism is an idempotent then that element is an idempotent. (2) If $[a]$ is an arbitrary element where $a \in S_i$. Then $a \leq g$ where g is invertible in S_i and so $[a] \leq [g]$ in S_∞ . But if g is invertible then $[g]$ is invertible. (3) Straightforward. (4) Straightforward. (5) This follows from the fact that, since the morphisms are all injective, the element $[a]$ is invertible if and only if a is invertible. \square

It follows that with each Bratteli diagram B we may associate a Boolean inverse \wedge -monoid constructed as a direct limit of semisimple inverse monoids and standard morphisms. We denote this inverse monoid by $\mathbf{l}(B)$.

Lemma 3.13. *Let*

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \dots$$

be a sequence of semisimple inverse monoids and injective morphisms. Then the direct limit $\varinjlim S_i$ is isomorphic to $I(B)$ for some Bratteli diagram B .

Proof. This follows by repeated application of Proposition 3.6. \square

We call any inverse monoid constructed in this fashion an *AF inverse monoid*. We may now summarize what we have found in this section in the following theorem.

Theorem 3.14. *AF inverse monoids are fundamental Foulis \wedge -monoids. Their groups of units are direct limits of finite direct products of finite symmetric groups where the morphisms between successive such direct products are by means of diagonal embeddings.*

The groups of units of AF inverse monoids are therefore the groups studied in [17, 35, 37].

3.2. An example. In this section, we shall construct a concrete example of an infinite MV-algebra that can be co-ordinatized by an inverse monoid. As we shall see, the monoid we construct is an analogue of the CAR algebra [51]. Recall that a non-negative rational number is said to be *dyadic* if it can be written in the form $\frac{a}{2^b}$ for some natural numbers a and b . The goal of the remainder of this section is to prove the following.

Theorem 3.15. *The MV-algebra of dyadic rationals in the closed unit interval $[0, 1]$ can be co-ordinatized by an inverse monoid.*

The inverse monoid in question will be what we term the dyadic inverse monoid. This will be constructed as a submonoid of the Cuntz inverse monoid which we describe first.

String theory

As a first step, we construct an inverse monoid, C_n , called the Cuntz inverse monoid. This was first described in [39, 40] but we have improved on the presentation given there and so we give it in some detail.

We begin by describing how we shall handle the Cantor space and its clopen subsets. Let A be a finite alphabet with n elements where $n \geq 2$. We shall primarily be interested in the case where $A = \{a, b\}$. We denote by A^* the set of all finite strings over A . The empty string is denoted by ε . We denote the total number of symbols occurring in the string x , counting repeats, by $|x|$. This is called the *length* of x . If $x, y \in A^*$ such that $x = yu$ for some finite string u , then we say that y is a *prefix* of x . We define $x \preceq y$ if and only if $x = yu$. This is a partial order on A^* called the *prefix order*. Observe that if $x \preceq y$ then x is at least as long as y . A pair of strings x and y are said to be *prefix comparable* if $x \preceq y$ or $y \preceq x$. A subset $X \subseteq A^*$ is called a *prefix subset* if for all $x, y \in X$ we have that $x \preceq y$ implies that $x = y$. If X is a prefix subset and contains the empty string then it contains only the empty string. If X is a prefix subset such that whenever $X \subseteq Y$, where Y is a prefix subset, we have that $X = Y$, then X is called a *maximal prefix subset*. Prefix subsets are often called *prefix codes*. We shall only consider *finite* prefix sets in this paper. If $X \subseteq A^*$ is a finite set, define $\max(X)$ to be the maximal elements of X under the prefix ordering. It is immediate that $\max(X)$ is a prefix set. We define the *length* of a prefix set X to be the maximum length of the strings belonging to X . We say that a prefix set X is *uniform of length l* if all strings in X have length l .

By A^ω we mean the set of all right-infinite strings over A . The set A^ω is equipped with the topology inherited from its representation as the space $A^\mathbb{N}$, where A is

given the discrete topology. It is the *Cantor space*. Up to homeomorphism, it is independent of the cardinality of A . Its clopen subsets are those subsets of the form XA^ω where $X \subseteq A^*$ is a finite set. The following result is well-known.

Lemma 3.16. *$xA^\omega \cap yA^\omega \neq \emptyset$ if and only if x and y are prefix comparable. If x and y are prefix-comparable, then either $xA^\omega \subseteq yA^\omega$ or $yA^\omega \subseteq xA^\omega$. In particular, if $x \preceq y$ then $xA^\omega \subseteq yA^\omega$.*

Proof. Suppose that $xA^\omega \cap yA^\omega \neq \emptyset$. Let $w \in xA^\omega \cap yA^\omega$. Then $w = xu = yv$ where u and v are infinite strings. If x and y have the same length, then $x = y$. Otherwise we may assume, without loss of generality, that $|x| > |y|$. It follows that y is a prefix of x and we can write $x = yc$ for some finite string c . Clearly, $xA^\omega \subseteq yA^\omega$. \square

It follows by the above lemma, that if $U = XA^\omega$ is a clopen set for some finite set X , then $U = \max(X)A^\omega$. Thus we may choose the set X to be a prefix set. This we shall always do from now on. If U is a clopen subset and $U = XA^\omega$, where X is a prefix set, then we say that X is a *generating set* of U . Observe that if $U = XA^\omega$ where X is a prefix set, then

$$U = \bigcup_{i=1}^m x_i A^\omega$$

is actually a disjoint union. The clopen subsets form a basis for the topology on the Cantor space. The sets XA^ω are called *cylinder sets*. Finite sets X will often be represented using the notation of *regular languages*. Thus if $X = \{x_1, \dots, x_m\}$, we shall also write $X = x_1 + \dots + x_m$. For more on infinite strings and proofs of any of the claims above, see [56].

Example 3.17. Let $A = \{a, b\}$. The representation of clopen subsets by prefix sets is not unique. For example, $aA^\omega = (aa + ab)A^\omega$, and $A^\omega = (a + b)A^\omega$.

The lack of uniqueness in the use of prefix sets to describe clopen subsets is something we shall have to handle. The next few results provide the means for doing so. We make no claims for originality, but include these results for the sake of clarity.

Lemma 3.18. *Let A be a finite alphabet and let X be a prefix set over A . Then $XA^\omega = X$ if, and only if, X is a maximal prefix set.*

Proof. Suppose first that X is a maximal prefix set of length l . Let w be any infinite string. Write $w = uw'$ where w' is infinite and u is the prefix of w of length l . The set $X + u$ properly contains X and so cannot be a prefix set. Thus u is prefix-comparable with an element of X . But, because of its length, it either equals an element of X or an element of X is a proper prefix of u . Thus there exists $x \in X$ such that $u = xu'$. It follows that $w = xu'w'$ and so $w \in XA^\omega$.

Conversely, suppose that $XA^\omega = X$. We prove that X is a maximal prefix set. Suppose not. Then there is at least one finite string u such that $X + u$ is a prefix set. Let w be any infinite string. Clearly, $uw \in A^\omega$. But then $uw = xw'$ where $x \in X$. Thus $uA^\omega \cap xA^\omega \neq \emptyset$. By Lemma 3.16, it follows that u and x are prefix-comparable, which is a contradiction. \square

We shall now describe two operations on a prefix set. In what follows, observe that for any $r \geq 0$, the set A^r is a maximal prefix set. The cases of interest below will always require $r \geq 1$. Let X be a prefix set. Let $u \in X$. Define

$$X^+ = (X - u) + uA^r,$$

where $r \geq 1$. We call X^+ an *extension* of X . Let $u \in X$ such that $uA^r \subseteq X$ for some $r \geq 1$. Define

$$X^- = (X - uA^r) + u.$$

We call X^- a *reduction* of X . The proof of the following is straightforward.

Lemma 3.19. *Let X be a prefix set. Then both X^+ and X^- are prefix sets and $XA^\omega = X^+A^\omega = X^-A^\omega$. In addition,*

$$X^{+-} = X \text{ and } X^{-+} = X.$$

Our next result shows that we may always replace a generating set by a uniform generating set.

Lemma 3.20. *Let $U = XA^\omega$ where X has length l . Then for each $r \geq l$ we may find a prefix set Y uniform of length r such that $U = YA^*$.*

Proof. Let $U = XA^\omega$ where X is a prefix set of length l . If all the strings in X have length l then we are done. Otherwise, let $u \in X$ such that $m = |u| < l$. Then by Lemma 3.19, we have that $X^+ = (X - u) + uA^{l-m}$ is a prefix set and that $XA^\omega = X^+A^\omega$. Thus the single string u has been replaced by $|A|$ strings each of length l . If all strings in X^+ have length l we are done, else we repeat the above procedure. In this way, we construct a prefix set X' uniform of length l such that $U = X'A^\omega$. It is now clear how this process can be repeated to obtain prefix sets generating U and uniform of any desired length $r \geq l$. \square

Example 3.21. Let $A = a + b$. Consider the clopen set $(aa + aba + b)A^\omega$. The length of $aa + aba + b$ is 3. Replace b by $b(a + b)^2$ and replace aa by $aa(a + b)$. We therefore get the prefix set

$$aa(a + b) + aba + b(a + b)^2$$

and we have, in addition, that

$$(aa + aba + b)A^\omega = (aa(a + b) + aba + b(a + b)^2)A^\omega.$$

Our next goal is to show that every clopen set has a ‘smallest’ generating set, in a suitable sense.

Lemma 3.22. *If $XA^\omega \subseteq YA^\omega$, where X and Y are prefix sets, then each element of X is a prefix comparable with an element of Y .*

Proof. Let $x \in X$. Then $xA^\omega \subseteq YA^\omega$. It follows that $xA^\omega = xA^\omega \cap YA^\omega$. Thus $xA^\omega = \bigcup_{y \in Y} xA^\omega \cap yA^\omega$. For at least one $y \in Y$, we must have that $xA^\omega \cap yA^\omega \neq \emptyset$. By Lemma 3.16, it follows that x and y are prefix-comparable. \square

The following is immediate by the above lemma.

Corollary 3.23. *Let $XA^\omega = YA^\omega$ where X and Y are prefix sets both uniform of the same length. Then $X = Y$.*

We define the *weight* of a prefix set X to be the sum $\sum_{x \in X} |x|$.

Lemma 3.24. *Let $U = XA^\omega = YA^\omega$ where X and Y have the same weight p . Suppose, in addition, that any generating set of U has weight at least p . Then $X = Y$.*

Proof. Let $x \in X$. By Lemma 3.22, there exists $y \in Y$ such that x and y are prefix-comparable. Suppose that $x \neq y$. Then, without loss of generality, we may assume that x is a proper prefix of y . Thus $y = xu$ for some finite string u . Consider the set $(Y - y) + x$. Observe that $U = ((Y - y) + x)A^\omega$. It is not possible for any element of $Y - y$ to be a prefix of x because then it would be a prefix of y which is a contradiction. It may happen that x is a prefix of some elements of $Y - y$.

So we consider $Y' = \max(Y - y)$. We have that Y' is a generating set of U and its weight is strictly less than p . This is a contradiction. We have therefore shown that if $x \in X$ then $x \in Y$. By symmetry, we deduce that $X = Y$. \square

Lemma 3.25. *Let $xA^\omega \subseteq YA^\omega$ where Y is a prefix set. Suppose that there is $y \in Y$ such that $y = xu$. then*

- (1) *If $y' \in Y$ then either $y'A^\omega \cap xA^\omega = \emptyset$ or $y' = xv$ for some v .*
- (2) *Denote by \bar{Y} the set of all elements of Y that have x as a prefix. Then $x^{-1}\bar{Y}$ is a maximal prefix set.*

Proof. (1) Suppose that $xA^\omega \cap xA^\omega \neq \emptyset$ where $y' \neq y$. If $x = y'v$ then y and y' are prefix-comparable, which is a contradiction. It follows that $y' = xv$.

(2) Suppose that $x^{-1}\bar{Y}$ is not a maximal prefix set. Let z be a string that is not prefix comparable with any string in $x^{-1}\bar{Y}$. Then xz is not prefix-comparable with any element of \bar{Y} . However, $xzA^\omega \subseteq xA^\omega$. Thus xz is prefix comparable with some element y'' of Y . If $y'' = xzz'$ for some z' then $y'' \in \bar{Y}$, which is a contradiction. Thus $xz = y''z'$. Thus x and y'' are prefix-comparable. By (1) above, we must have that $y'' = xd$ for some string d . Thus $d \in x^{-1}\bar{Y}$. But $xz = xdz'$. Thus $z = dz'$. But this is a contradiction. \square

Lemma 3.26. *Let X be a prefix set. Suppose that $xZ \subseteq X$ where Z is a maximal prefix set, where $Z \neq \emptyset$. Then it is possible to apply reduction to X .*

Proof. It is enough to show that we may apply reduction to Z . Let $z \in Z$ be a string of maximal length. Suppose that $z = z'a$ where $a \in A$. I claim that $z'A \subseteq Z$. Let $b \in A$ where $b \neq a$. Then $z'b$ is a string the same length as z . So it too has maximal length. Since Z is a maximal prefix set, it follows that $z'b$ must be prefix-comparable with some element of Z . So it either belongs to Z , and we are done, or some element of Z of length at least one less is a prefix of $z'b$, which is impossible. \square

Proposition 3.27. *Let $U = XA^\omega$. Construct the prefix code X' from X by carrying out any sequence of reductions until this is no longer possible. Then X' is a generating set of U of minimum weight.*

Proof. The fact that X' is a generating set follows by Lemma 3.19. Suppose that $U = YA^\omega$ where Y has strictly smaller weight than X . Let $x \in X'$. Then by Lemma 3.22, x must be prefix-comparable with some $y \in Y$. Suppose for each $x \in X'$, it were the case that there was an element $y_x \in Y$ such that x was a prefix of y_x . If $x, x' \in X'$ were both prefixes of y , then they would have to be prefix-comparable. It would then follow that the weight of Y was equal to or greater than the weight of X' , which is a contradiction. Since the weights of the two prefix sets are different the sets cannot be equal. It follows that there is at least one $x \in X'$ and $y \in Y$ such that $x = yu$ for some finite string u of length $r \geq 1$. We have that $yA^\omega \subseteq X'A^\omega$. By Lemma 3.25, it follows that all the elements of X' that have y as a prefix forms a subset yZ where Z is a maximal prefix code. Then by Lemma 3.26, it is possible to apply a reduction to X' , which is a contradiction. \square

By Proposition 3.27 and Lemma 3.24, it follows that every clopen subset is generated by a unique prefix set of minimum weight. We call this the *minimum generating set*.

Lemma 3.28. *Suppose that $U = XA^\omega = YA^\omega$. Then Y is obtained from X by a finite sequence of extensions and reductions.*

Proof. By means of a sequence of reductions X may be converted to the minimum generating set X_U by Lemma 3.27. Likewise Y may be converted to the minimum

generating set X_U . Starting with X_U we may therefore construct Y by a sequence of extensions applying Lemma 3.19. Combing these two sequences together we may convert X to Y . \square

The Cuntz inverse monoid

We can now set about constructing an inverse monoid. Let $A = a_1 + \dots + a_n$, though in the case $n = 2$, we shall usually assume that $A = a + b$. The *polycyclic monoid* P_n , where $n \geq 2$, is defined as a monoid with zero by the following presentation

$$P_n = \langle a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1} : a_i^{-1}a_i = 1 \text{ and } a_i^{-1}a_j = 0, i \neq j \rangle.$$

It is, in fact, an inverse monoid with zero. Every non-zero element of P_n is of the form yx^{-1} where $x, y \in A_n^*$, and where we identify the identity with the element $1 = \varepsilon\varepsilon^{-1}$. The product of two elements yx^{-1} and vu^{-1} is zero unless x and v are prefix-comparable. If they are prefix-comparable then

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{if } v = xz \text{ for some string } z \\ y(uz)^{-1} & \text{if } x = vz \text{ for some string } z \end{cases}$$

The non-zero idempotents in P_n are the elements of the form xx^{-1} , where x is positive, and the natural partial order is given by $yx^{-1} \leq vu^{-1}$ iff $(y, x) = (v, u)p$ for some positive string p . See [38, 39, 40] for more about the polycyclic inverse monoids, and proofs of any claims.

We may obtain an isomorphic copy of P_n as an inverse submonoid of $I(A^\omega)$ as follows. Let $yx^{-1} \in P_n$. Then define a map from xA^ω to yA^ω by $xw \mapsto yw$ where w is any right-infinite string. Thus yx^{-1} describes the process *pop the string* x and then *push the string* y .

Remark 3.29. In what follows, we shall always regard P_n as an inverse submonoid of $I(A^\omega)$.

We now construct a larger inverse monoid containing this copy of P_n . The inverse monoid $I(A^\omega)$ is a Boolean inverse monoid. Thus finite non-empty compatible subsets have joins. Let $S \subseteq I(A^\omega)$ be an inverse submonoid containing zero. Then we may form the subset S^\vee consisting of all joins of finite non-empty compatible subsets of S . It is routine to check that S^\vee is again an inverse submonoid of $I(A^\omega)$. We apply this construction to P_n to obtain the inverse submonoid P_n^\vee .

Lemma 3.30. *Let yx^{-1} and vu^{-1} be a compatible pair of elements in the polycyclic inverse monoid P_n . If they are not orthogonal, then either $yx^{-1} \leq vu^{-1}$ or vice-versa.*

Proof. Without loss of generality, suppose that $xy^{-1}vu^{-1} \neq 0$. Then y and v are prefix-comparable. Again, without loss of generality, we may assume that $y = vz$ for some z . Then $xy^{-1}vu^{-1} = x(uz)^{-1}$. But this is supposed to be an idempotent and so $x = uz$. We substitute this into $yx^{-1}uv^{-1}$ to get $y(vz)^{-1}$. But this too is supposed to be an idempotent and so $y = vz$. We have therefore proved that $yx^{-1} \leq vu^{-1}$. \square

From the above lemma, a finite non-empty compatible subset of P_n will have the same join as a finite non-empty orthogonal subset obtained by taking the maximal elements of the compatible subset.

Lemma 3.31. *A subset*

$$\{y_1x_1^{-1}, \dots, y_mx_m^{-1}\}$$

of P_n is orthogonal iff $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_m\}$ are both prefix sets.

It follows that the elements of P_n^\vee can be represented in the following form. Let $x_1 + \dots + x_r$ and $y_1 + \dots + y_r$ be two prefix sets with the same number of elements. Define a map from $(x_1 + \dots + x_r)A^\omega$ to $(y_1 + \dots + y_r)A^\omega$, denoted by,

$$\begin{pmatrix} x_1 & \dots & x_r \\ y_1 & \dots & y_r \end{pmatrix}$$

that does the following: $x_i w \mapsto y_i w$, where w is any right-infinite string. We denote the totality of such maps by $C_n = P_n^\vee$. We call this the *Cuntz inverse monoid (of degree n)*. We shall call the unique countable atomless Boolean algebra the *Tarski algebra*. The following was proved in [40]. But we shall give the details below. Recall that an inverse semigroup with zero is *0-simple* if there are only two ideals. It is well-known that a 0-simple, fundamental Boolean inverse monoid is congruence-free.

Proposition 3.32. *C_n is a Boolean inverse \wedge -monoid whose semilattice of idempotents is the Tarski algebra. It is fundamental, 0-simple and has n \mathcal{D} -classes. It is therefore congruence-free. Its group of units is the Thompson group V_n .*

Proof. We show first that we have a groupoid. Suppose that $f: XA^\omega \rightarrow YA^\omega$ be such that there is a bijection $f_1: X \rightarrow Y$ such that $f(xw) = f_1(x)w$ for any infinite string w . Suppose that $X^+ = x_1A + x_2A + \dots + x_rA$. Let $Y^+ = y_1A + y_2A + \dots + y_rA$. Define $f^+: X^+ \rightarrow Y^+$ as follows. Let $f^+(x_i) = y_i$ for $2 \leq i \leq r$. Define $f^+(x_1a_j) = y_1a_j$ for $1 \leq j \leq n$. It is clear that $f^+ = f$. We shall call it a *refinement* of f . Let $g: UA^\omega \rightarrow VA^\omega$ and suppose that $XA^\omega = VA^\omega$. Let X' be obtained from X by a sequence of extensions. let Y' be obtained from Y by a sequence of extensions. By Lemma 3.20, we suppose that X' and Y' are both uniform of the same length. We construct Y' and U' by using the corresponding extensions. It follows by Corollary 3.23 that $X' = Y'$. Let f'_1 be obtained from f_1 by successive appropriate refinements, and likewise let g'_1 be obtained from g_1 . Thus $f' = f$ and $g' = g$. But we may now compose $f'_1g'_1$ directly to get a map from U' to Y' that represents fg . Since inverses pose no problems, we have shown that we have a groupoid. The semilattice of idempotents is just the Tarski algebra. We show that this is an ordered groupoid, and so inductive, from which we get that it is an inverse monoid. Let $f: XA^\omega \rightarrow YA^\omega$ where $f_1: X \rightarrow Y$ is a bijection. Let $ZA^\omega \subseteq XA^\omega$. Assume first that each element $z \in Z$ can be written $z = xu$ for some $x \in X$ and string u . Observe that under this assumption, x will be unique. Define $g_1(z) = f_1(x)u$. Put Y' equal to the set of all $f_1(x)u$ as $z \in Z$. Then $Y'A^\omega \subseteq YA^\omega$ and we have defined a bijection $g: ZA^\omega \rightarrow Y'A^\omega$ which is the restriction of f . It remains to show that we can verify our assumption. This can be achieved as in Lemma 3.20 by using a sequence of extensions to convert Z into a prefix set where all strings have lengths strictly larger than the longest string in X . Then by Lemma 3.22, since $ZA^\omega \subseteq XA^\omega$, we have that each element of Z is prefix-comparable with an element of X . From length considerations, it follows that each $z \in Z$ has as a prefix an element of X .

It is straightforward to see that C_n is a Boolean inverse monoid and that it is also a \wedge -monoid.

We now prove that C_n is 0-simple. Let X and Y be any two prefix sets. let $y \in Y$. Then yX is a prefix set with the same cardinality as X . It follows that there is an element $f: XA^\omega \rightarrow yXA^\omega$ of C_n . But $yXA^\omega \subseteq YA^\omega$. This proves the claim.

We now prove that there are $n-1$ non-zero \mathcal{D} -classes. The first step is to calculate the number of strings in a *maximal* prefix set. For a fixed $n \geq 2$, and for $r = 0, 1, 2, \dots$, we can construct maximal prefix sets containing $P_r^n = (r-1)n - (r-2)$ strings. Concrete examples of such sets can be constructed by starting with the

‘seeds’ ε and A and then growing maximal prefix sets by attaching A from left-to-right. We designate these specific maximal prefix sets by M_r^n . There are $n - 2$ numbers between P_r^n and P_{r+1}^n . Consider now the $n - 2$ prefix sets $C_1^n = a_1 + a_2$, $C_2^n = a_1 + a_2 + a_3$, \dots , $C_{n-2}^n = a_1 + \dots + a_{n-1}$. The partial identities associated with C_i and C_j where $i \neq j$ are not \mathcal{D} -related. There are therefore at least n \mathcal{D} -classes when we add in the zero and the identity. We may attach a copy of M_r^n to the rightmost vertex of C_i . We denote this prefix set by $C_i * M_r^n$. Observe that $C_i A^\omega = C_i * M_r^n$. Let X be an arbitrary prefix set. Either it is in bijective correspondence with one of the M_r^n , in which case the identity function on XA^ω is \mathcal{D} -related to the identity, or it is in bijective correspondence with one of the $C_i * M_r^n$, in which case the identity function on XA^ω is \mathcal{D} -related to the identity function on C_i . In particular, we see that C_2 is bisimple.

The group of units of C_n consists of those elements

$$\begin{pmatrix} x_1 & \dots & x_r \\ y_1 & \dots & y_r \end{pmatrix}$$

where $x_1 + \dots + x_r$ and $y_1 + \dots + y_r$ are maximal prefix codes. These are precisely the elements of Thompson’s group V_n . \square

The dyadic (or CAR) inverse monoid

We shall need to work with measures on the Cantor set. The general theory of such measures is the subject of current research, see [1, 2, 3, 10], for example but the measures we need are well-known.

Let A be an alphabet with n elements. Define $\mu(a) = \frac{1}{n}$ for any $a \in A$ and define $\mu(\varepsilon) = 1$. If $x \in A^*$ is any string of length r define $\mu(x) = \frac{1}{n^r}$. If X is any prefix set, define $\mu(X) = \sum_{x \in X} \mu(x)$. The following is proved as [56, Theorem I.4.2].

Lemma 3.33. *For any prefix set X , we have that $\mu(X) \leq 1$.*

Let U be any clopen subset of A^ω . Suppose that $U = XA^\omega$. Define $\mu(U) = \mu(X)$. We call μ defined in this way on the clopen subsets of A^ω the *Bernoulli measure*. This measure is sometimes denoted $\beta(\frac{1}{n})$.

Lemma 3.34.

- (1) *Let X be a prefix set. Then $\mu(X) = 1$ if, and only if X is a maximal prefix set.*
- (2) *The Bernoulli measure is well-defined.*

Proof. (1) let X be a maximal prefix set. It is obtained by means of a sequence of extensions from ε and $\mu(\varepsilon) = 1$. Clearly, $\mu(A^r) = 1$. Thus if Y_1 and Y_2 are prefix sets and Y_2 is an extension of Y_1 then $\mu(Y_2) = \mu(Y_1)$. The result follows. Suppose now that $\mu(X) = 1$. If X is not maximal, then we can find a string u such that $X + u$ is a prefix set. But $\mu(X + u) = \mu(X) + \mu(u) > 1$, which is a contradiction.

(2) This follows by Proposition 3.27. \square

The following result will be important later.

Lemma 3.35. *Let A be an alphabet with $n \geq 2$ elements. Let $U = XA^\omega$ and $V = YA^\omega$ be such that X has length l , and Y has length m . Without loss of generality, we may assume that $m \geq l$. Suppose that $\mu(U) = \mu(V)$. Then there is a prefix set X' uniform of length m such that $U = X'A^\omega$, and there is a prefix set Y' uniform of length m such that $V = Y'A^\omega$, such that $|X'| = |Y'|$.*

Proof. By Lemma 3.20, we may find a prefix set X' , uniform of length m , such that $U = X'A^\omega$. Observe that $\mu(X) = \mu(X')$. Let r be the number of strings in X' . Then $\mu(X) = \frac{r}{n^m}$. Similarly, we may find a prefix set Y' , uniform of length m ,

such that $V = Y'A^\omega$. Observe that $\mu(Y) = \mu(Y')$. Let s be the number of strings in Y' . Then $\mu(Y) = \frac{s}{n^r}$. It follows immediately that $r = s$, as required. \square

The following result was first proved in [41] but suggested by earlier work of Meakin and Sapir [49]. It shows how to construct inverse submonoids of the polycyclic inverse monoid. A *wide* inverse subsemigroup of S is one that contains all the idempotents of S .

Proposition 3.36. *Let A be an n -letter alphabet. Then there is a bijection between right congruences on A^* and wide inverse submonoids of P_n . If ρ is the right congruence in question, then the corresponding inverse submonoid of P_n simply consists of 0 and all elements yx^{-1} where $(y, x) \in \rho$.*

Consider now the congruence λ of the length map $A^* \rightarrow \mathbb{N}$ given by $x \mapsto |x|$. Define $G_n \subseteq P_n$ to consist of zero and all elements yx^{-1} where $|y| = |x|$. Then by Proposition 3.36, G_n is an inverse monoid. It was first defined in the thesis of David Jones [31] and is called the *gauge inverse monoid (on n letters)* and arose from investigations of strong representations of the polycyclic inverse monoids [32] motivated by the theory developed in [13].

We now define $Ad_n \subseteq C_n$, called the *n -adic inverse monoid*. In the case $n = 2$, we refer to the *dyadic inverse monoid*. By definition, it consists of those elements of C_n which are orthogonal joins of elements of G_n . That is, maps of the form

$$\begin{pmatrix} x_1 & \dots & x_r \\ y_1 & \dots & y_r \end{pmatrix}$$

where $|y_i| = |x_i|$ for $1 \leq i \leq r$. The proof of the following is immediate.

Proposition 3.37. *The n -adic inverse monoid is a fundamental Boolean inverse monoid and wide inverse submonoid of the Cuntz inverse monoid C_n .*

The following result will establish most of the properties we shall need to prove our main theorem.

Proposition 3.38. *The dyadic inverse monoid may be equipped with a good invariant mean that reflects the \mathcal{D} -relation.*

Proof. The idempotents of A_2 are simply the clopen subsets of the Cantor space. We equip these with the Bernoulli measure $\beta(\frac{1}{2})$. We show first that μ is an invariant mean. There is only one property we have to check. Let \mathbf{e} and \mathbf{f} be \mathcal{D} -related idempotents in A_2 . Let \mathbf{e} be the identity function on the clopen subset U and let \mathbf{f} be the identity function on the clopen subset V . Then there are prefix sets $X = x_1 + \dots + x_r$ and $Y = y_1 + \dots + y_r$ such that $U = (x_1 + \dots + x_r)A^\omega$ and $V = (y_1 + \dots + y_r)A^\omega$ such that $y_i x_i^{-1}$ are elements of the gauge inverse monoid. That is, we have a map

$$\begin{pmatrix} x_1 & \dots & x_r \\ y_1 & \dots & y_r \end{pmatrix}$$

where $|y_i| = |x_i|$ for $1 \leq i \leq r$ from \mathbf{e} to \mathbf{f} . In particular, the sets X and Y contain the same number of strings, and the same number of strings of the same length. It is now immediate that $\mu(\mathbf{e}) = \mu(\mathbf{f})$. The fact that the \mathcal{D} -relation is reflected follows from Lemma 3.35.

It remains to prove that this invariant mean is good. Let $\mu(\mathbf{e}) \leq \mu(\mathbf{f})$. We work with clopen subsets and so we assume that $\mu(U) \leq \mu(V)$. This may be easily deduced using Lemma 3.20 and a modified version of Lemma 3.35. \square

The above proposition, combined with Lemma 2.8, tells us that the dyadic inverse monoid is a Foulis monoid and that its lattice of principal ideals forms a linearly

ordered set isomorphic to the dyadic rationals in the unit interval. We therefore now have the main result of this section.

Theorem 3.39. *The MV-algebra of dyadic rationals is co-ordinatized by the dyadic inverse monoid.*

It is worth looking in more detail at the structure of the dyadic inverse monoid Ad_2 .

Proposition 3.40. *The dyadic inverse monoid is isomorphic to the direct limit of the sequence*

$$I_1 \rightarrow I_2 \rightarrow I_4 \rightarrow I_8 \rightarrow \dots$$

It is therefore an AF inverse monoid.

Proof. Let $A = a + b$. We construct the binary tree with root A^ω and then vertices aA^ω and bA^ω at the first level, aaA^ω , abA^ω , baA^ω and bbA^ω at the second level, and so on. The clopen sets at each level are pairwise disjoint. Every clopen set has a generating set constructed from taking the union of the above sets at the same level. This is a result of Lemma 3.20. However, the same subset can, of course, be represented in different ways. Thus the clopen set aA^ω which is from level 1, can also be written as $aaA^\omega + abA^\omega$, a union of sets constructed from level 2.

We now observe that the elements of Ad_2 constructed from the gauge inverse monoid maps at level l form an inverse monoid isomorphic to I_{2^l} . The best way to see this is that at level l we may construct all the relevant matrix units together with the identity and the zero. For example, at level 2, we have, in addition to the identity and the zero, the 4 idempotents

$$aa(aa)^{-1}, ab(ab)^{-1}, ba(ba)^{-1}, bb(bb)^{-1}$$

and then the non-identity matrix units such as $aa(ab)^{-1}$. By taking joins we get all the other elements of I_{2^l} . In addition, we see that this copy is actually an inverse submonoid of Ad_2 containing the zero.

We claim next that $I_{2^l} \subseteq I_{2^{l+1}}$. This is also best seen by focusing on the matrix units. First observe that every idempotent of level l is also an idempotent at level $l + 1$. If XA^ω is a clopen subset with X a union of idempotents at level l then $XA^\omega = XaA^\omega + XbA^\omega$. It follows that every element of I_l reappears in I_{l+1} by the process of refinement.

It is now evident that $Ad_2 = \bigcup_{l=1}^{\infty} I_{2^l}$, which proves the theorem. \square

Remark 3.41. In the light of the above result, we might also call the dyadic inverse monoid the *CAR inverse monoid*.

The group of units of Ad_2 is the direct limit $S_1 \rightarrow S_2 \rightarrow S_4 \rightarrow \dots$ where the inclusions between successive symmetric groups are block diagonal maps.

4. PROOF OF THE MAIN THEOREM

The goal of this section is to prove the following.

Theorem 4.1 (Co-ordinatization). *Let E be a countable MV-algebra. Then there is a Foulis monoid S satisfying the lattice condition such that S/\mathcal{I} is isomorphic to E .*

We begin by giving some standard definitions and results we shall need.

An *ordered abelian group* G is given by a submonoid $G^+ \subseteq G$ called the *positive cone* such that $G^+ \cap (-G^+) = \{0\}$ and $G = G^+ - G^+$. If $a, b \in G$ define $a \leq b$ if, and only if, $b - a \in G^+$. The condition $G = G^+ - G^+$ means that G is the group of fractions of its positive cone. The condition $G^+ \cap (-G^+) = \{0\}$ means that 0 is the only invertible element of G^+ . We say that G^+ is *conical* if it has trivial units. The

theory of abelian monoids tells us that every abelian conical cancellative monoid arises as the positive cone of an ordered abelian group. If the order in a partially ordered abelian group G actually induces a lattice structure on G we say that the group is *lattice-ordered* or an *l-group*.

Let G be a partially ordered abelian group. An *order unit* is a positive element u such that for any $g \in G$ there exists a natural number n such that $g \leq nu$. Let $u \in G$ be any positive, non-zero element. Denote by $[0, u]$ the set of all elements g such that $0 \leq g \leq u$. The notation is not intended to suggest that this set is linearly ordered. Let $p, q \in [0, u]$. Define the partial binary operation \oplus on $[0, u]$ by $p \oplus q = p + q$ if $p + q \in [0, u]$, and undefined otherwise. If $p \in [0, u]$ define $p' = u - p$. Then $[0, u]$ becomes an effect algebra [25, Theorem 3.3]. We call this the *interval effect algebra* associated with (G, u) . If, in addition, G is an *l-group* and u is an order-unit, then $[0, u]$ is actually an MV-algebra. The following is proved in [50, Theorem 3.9], [16, Corollary 7.1.8] and [52].

Theorem 4.2. *Every MV-algebra is isomorphic to an interval effect algebra $[0, u]$ where u is an order unit in an *l-group*.*

We briefly sketch out how the above theorem may be proved. If $(E, \oplus, 0)$ is a partial algebra, then we may construct its *universal monoid* $\nu: E \rightarrow M_E$ in the usual way. However, we are interested not merely in the existence of M_E but in its properties so we shall give more details on how the universal monoid is constructed. The proof of part (1) below follows from [6] and [18, Lemma 1.7.6, Proposition 1.7.7, Proposition 1.7.8, Lemma 1.7.10, Lemma 1.7.11, Theorem 1.7.12]. It is noteworthy that commutativity arises naturally and does not have to be imposed. The proof of part (2) below follows from [18, Theorem 1.7.12]. Alternative approaches can be found in [27, 63].

Proposition 4.3. *Let $(E, \oplus, 0)$ be a conical partial refinement monoid.*

- (1) *Let E^+ denote the free semigroup on E . Define \sim to be the congruence on E^+ generated by $(a, b) \sim (a \oplus b)$ when $\exists a \oplus b$. Put $M = E^+ / \sim$. Then M is a conical abelian monoid and is the universal monoid of E .*
- (2) *Suppose that $(E, \oplus, 0, 1)$ is also an effect algebra. Then M is cancellative, the image of E in M is convex, and the image of 1 in M is an order unit.*

An abelian monoid always has a universal group: its *Grothendieck group*. If the abelian monoid is cancellative and conical then its Grothendieck group is partially ordered and is its group of fractions. It follows that the Grothendieck group of the universal monoid of an effect algebra satisfying the refinement property is the *universal group* of that effect algebra. This leads to the main theorem we shall need proved by Ravindran [59]. Its full proof may be found as [18, Theorem 1.7.17].

Theorem 4.4 (Ravindran). *Let E be an effect algebra satisfying the refinement property.*

- (1) *The universal group $\gamma: E \rightarrow G_E$ is a partially ordered abelian group with the refinement property. Its positive cone P is generated as a submonoid by the image of E under γ .*
- (2) *Put $u = \gamma(1)$. Then u is an order unit in G_E and E is isomorphic under γ to the interval effect algebra $[0, u]$.*
- (3) *If E is actually an MV-algebra, then $[0, u]$ is a lattice from which it follows that G_E is an *l-group*. If E is countable then G_E is countable.*

The proof of the following is immediate but it is significant from the point of view of the main goal of this paper.

Proposition 4.5. *Let S be a Foulis monoid. Then S/\mathcal{J} is isomorphic to the interval $[0, u]$ where u is an order unit in the universal group of $E(S)$ and is the image of the class of the identity of $E(S)$.*

Every AF inverse monoid is a Foulis monoid by Theorem 3.14. Accordingly, our first aim will be to explicitly compute the universal group of the effect algebra associated with an AF inverse monoid. To do this, it will be useful to work with the idempotents of the inverse monoid directly rather than with the elements of the associated effect algebra. This is the import of the following definition.

Let S be a Boolean inverse monoid. A *group-valued invariant mean* on S is a function $\theta: E(S) \rightarrow G$ to an abelian group G such that the following two axioms hold:

(GVIM1): If e and f are orthogonal then $\theta(e \vee f) = \theta(e) + \theta(f)$.

(GVIM2): We have that $\theta(s^{-1}s) = \theta(ss^{-1})$ for all $s \in S$.

It follows from (GVIM1) that $\theta(0) = 0$. By the usual considerations, a universal group-valued invariant mean always exists.

The following lemma tells us that we can, indeed, pull-back to the set of idempotents of the inverse monoid.

Lemma 4.6. *Let S be a Foulis monoid. Then the universal group-valued invariant mean is the universal group of the associated effect algebra.*

Proof. Let $\nu: E(S) \rightarrow G_S$ be the universal group-valued invariant mean. Denote by $\nu': E(S) \rightarrow G_S$ by $\nu'([e]) = [\nu(e)]$. This is a well-defined map such that if $[e] \oplus [f]$ exists then $\nu'([e] + [f]) = \nu'([e]) + \nu'([f])$. Because of axiom (GVIM2), we may define a function $\mu: E(S) \rightarrow G_S$ by $\mu([e]) = \nu(e)$. Suppose that $[e] \oplus [f]$ is defined. Then it equals $[e' \vee f']$ where $e \mathcal{D} e'$ and $f \mathcal{D} f'$. But $\nu(e' \vee f') = \nu(e') + \nu(f')$, and so $\mu([e]) + \mu([f]) = \mu([e] \oplus [f])$.

Now let $\theta: E(S) \rightarrow H$ be any map to a group such that if $[e] \oplus [f]$ is defined then $\theta([e] \oplus [f]) = \theta([e]) + \theta([f])$. Define $\phi: E(S) \rightarrow H$ by $\phi(e) = \theta([e])$. Then it is immediate that ϕ is a group-valued invariant mean. It follows that there is a group homomorphism $\alpha: G_S \rightarrow H$ such that $\alpha\nu = \phi$. Clearly, $\alpha\nu' = \theta$. \square

We now set about computing the universal group-valued mean of an AF inverse monoid. First, we shall need some definitions. A *simplicial group* is simply a group of the form \mathbb{Z}^r with the usual ordering. A *positive homomorphism* between simplicially ordered groups maps positive elements to positive elements. If the ordered groups are also equipped with distinguished order units, then a homomorphism is said to be *normalized* if it maps distinguished order units one to the other. A *dimension group* is defined to be a direct limit of a sequence of simplicially ordered groups and positive homomorphisms. An ordered abelian group is said to satisfy the *Riesz interpolation property* (RIP) if $a_1, a_2 \leq b_1, b_2$, in all possible ways, implies that there is an element c such that $a_1, a_2 \leq c$ and $c \leq b_1, b_2$. Such a group satisfies the *Riesz decomposition property* (RDP) if for all *positive* a, b, c if $a \leq b + c$ implies that there are positive elements b', c' such that $b' \leq b$ and $c' \leq c$ and $a = b' + c'$. These two properties (RIP and RDP) are equivalent for partially ordered abelian groups [26, Proposition 21.3] (but not for effect algebras). The partially ordered abelian group (G, G^+) is said to be *unperforated* if $g \in G$ and $ng \in G^+$ for some natural number $n \geq 1$ implies that $g \in G^+$. The proof of part (1) of the following is part of [19, Theorem 3.1], and the proof of part (2) is from [26, Corollary 21.9]

Theorem 4.7.

- (1) *Countable partially ordered abelian groups are dimension groups precisely when they satisfy the Riesz interpolation property and are unperforated.*

- (2) Each countable dimension group with a distinguished order unit is isomorphic to a direct limit of a sequence of simplicial groups with order-units and normalized positive homomorphisms. Thus each such group is constructed from a Bratteli diagram.
- (3) Countable l -groups are dimension groups.

We may now explicitly compute the universal group-valued invariant means of AF inverse monoids. We begin with a special case. In what follows, we denote by $|e|$ the cardinality of the set A where $e = 1_A$.

Lemma 4.8.

- (1) Let I_n be a finite symmetric inverse monoid on n letters. Define the function $\pi: E(I_n) \rightarrow \mathbb{Z}$ by $\pi(1_A) = |A|$. Then π is the universal group-valued invariant mean of I_n and the image of the identity is n , an order unit.
- (2) Let $T = S_1 \times \dots \times S_r$ be a semisimple inverse monoid, where $n(1), \dots, n(r)$ are the number of letters in the underlying sets of S_1, \dots, S_r , respectively. Put $\mathbf{n} = (n(1), \dots, n(r))$. Define

$$\pi: E(S_1 \times \dots \times S_r) \rightarrow \mathbb{Z}^r$$

by

$$\pi(e_1, \dots, e_r) = (|e_1|, \dots, |e_r|).$$

Then π is the universal group-valued invariant mean of T and the identity of T is mapped to the order unit \mathbf{n} .

Proof. (1) It is straightforward to check that π has the requisite properties. The universal property follows from the fact that the atoms of $E(I_n)$ are mapped to the identity of \mathbb{Z} . The proof of (2) follows from (1). \square

We may now prove the general case.

Proposition 4.9. Let B be a Bratteli diagram with associated AF inverse monoid $l(B)$ and associated dimension group $G(B)$. Then the universal group-valued invariant mean of $l(B)$ is given by a map $\pi: E(l(B)) \rightarrow G(B)$ where the image of the identity of $E(l(B))$ is an order unit u in $G(B)$.

Proof. From the Bratteli diagram B , we may construct a sequence

$$T_0 \xrightarrow{\sigma_0} T_1 \xrightarrow{\sigma_1} T_2 \xrightarrow{\sigma_2} \dots$$

of semisimple inverse monoids and injective standard morphisms. By definition, $l(B) = \varinjlim T_i$. Observe that $E(l(B)) = \varinjlim E(T_i)$. We begin by defining a map $\pi: E(l(B)) \rightarrow G(B)$, that will turn out to have the required properties. We consider level i of the Bratteli diagram B . The semisimple inverse monoid T_i is a product $S_1 \times \dots \times S_{r(i)}$ of $r(i)$ symmetric inverse monoids, where $n(1), \dots, n(i)$ is the number of letters in the underlying sets of $S_1, \dots, S_{r(i)}$, respectively. Put $\mathbf{n}(i) = (n(1), \dots, n(i))$. Define

$$\pi_i: E(S_1 \times \dots \times S_{r(i)}) \rightarrow \mathbb{Z}^{r(i)}$$

as in Lemma 4.8. Then also by Lemma 4.8, $\pi_i: E(T_i) \rightarrow \mathbb{Z}^{r(i)}$ is the universal group-valued invariant mean of T_i and the identity of T_i is mapped to the order unit $\mathbf{n}(i)$. Let $\beta_i: \mathbb{Z}^{r(i)} \rightarrow \mathbb{Z}^{r(i+1)}$ be the $r(i+1) \times r(i)$ matrix defined after Remark 3.5. We also denote by σ_i the restriction of that map to $E(T_i)$. We claim that $\beta_i \pi_i = \pi_{i+1} \sigma_i$ and that it is a normalized positive homomorphism. This follows from two special cases. First, we consider the standard map from R_m to R_n given by $A \mapsto sA$. If A represents an idempotent then $|A|$ is simply the number of 1's along the diagonal. Clearly, $|sA| = s|A|$. Thus the corresponding map β from \mathbb{Z} to \mathbb{Z} is simply multiplication by s . Observe that $sm = n$. Second, we consider the standard

map from $R_{m(1)} \times \dots \times R_{m(k)}$ to R_n given by $(A_1, \dots, A_k) \mapsto s_{i1}A_1 \oplus \dots \oplus s_{ik}A_k$ where $n = s_1m(1) + \dots + s_km(k)$. The corresponding map from $\mathbb{Z}^k \rightarrow \mathbb{Z}$ is given by the $1 \times k$ -matrix

$$(s_1 \quad \dots \quad s_k)$$

Our claim now follows. Thus from the properties of direct limits that we have a well-defined map $\pi: E(\mathbf{l}(B)) \rightarrow \mathbf{G}(B)$, by construction it is a group-valued invariant mean, and the image of the identity is an order-unit. The fact that it has the requisite universal properties follows from the fact that each map π_i has the requisite universal properties. \square

The following theorem combines Proposition 4.5, Proposition 4.9 and Theorem 4.4 in the form that we shall need.

Theorem 4.10. *Let S be an AF inverse monoid satisfying the lattice condition. Then the universal group-valued invariant mean $\mu: E(S) \rightarrow G_S$ is such that G_S is a countable l -group and the image of the identity of S in G_S is an order unit u . In addition, S/\mathcal{I} is isomorphic to $[0, u]$ as an MV-algebra.*

We may now prove Theorem 4.1. Let E be a countable MV-algebra. Then by Theorem 4.2 and Theorem 4.4, E is isomorphic to the MV-algebra $[0, u]$ where u is an order-unit in the universal group G of E . The group G is a countable l -group and by Theorem 4.7, it is a countable dimension group. Thus there is a Bratteli diagram B such that $\mathbf{G}(B) = G$. Let $\mathbf{l}(B)$ be the AF inverse monoid constructed from B . Then by Proposition 4.9 and Theorem 4.10, we have that $\mathbf{l}(B)/\mathcal{I}$ is isomorphic to $[0, u]$ as an MV-algebra. Observe that $\mathbf{l}(B)/\mathcal{I}$ satisfies the lattice condition, because $[0, u]$ is a lattice. It follows that we have co-ordinatized the MV-algebra E by means of the AF inverse monoid that satisfies the lattice condition.

5. CONCLUDING REMARKS

In this paper, we have shown how to co-ordinatize all countable MV-algebras, and concretely illustrated the result with the construction of the dyadic inverse monoid. We leave to future work the problem of constructing concrete examples of inverse monoids that co-ordinatize well-known countable MV-algebras such as the rationals and algebraic numbers in $[0, 1]$, as well as the free MV-algebras on finitely many generators. For a long list of examples of countable MV-algebras, see Table 1 of Mundici [51]. Our theory is adapted to working with the countable case only. This leaves completely open the question of uncountable cardinalities as well as the still more general question of co-ordinatizing effect algebras.

The theory of effect algebras once seemed like a niche area of research in mathematics, but recent work has suggested that it may occupy a more central position. In particular, the work of Jacobs [28] illustrates the breadth and scope of effect algebras, while suggesting a framework for understanding those categories which admit some kind of dimension theory. For a general lattice-theoretic treatment of dimension theory, generalizing the work of von Neumann, see Wehrung [63], and for some preliminary connections of effect algebras to traditional dimension groups, see [30]. It is too early to say how our work and that of Jacobs are related, though we might speculate that it occupies a position midway between his categories and the effect algebras. Specifically, our work should be generalizable to inverse categories and this might lead to some insight into the connections.

Finally, it is noteworthy that Elliott's original construction of what he calls the local semigroup associated with a C^* -algebra [20], which the main construction of our paper parallels, is, in fact, the construction of an effect algebra. This raises the question of whether effect algebras have the potential to provide a finer class of invariants for C^* -algebras.

6. APPENDIX: AF INVERSE MONOIDS AND AF C^* ALGEBRAS

This section is not needed to prove our main results. Instead, it is intended to show that there is a closer connection between AF inverse monoids and AF C^* -algebras than merely one of analogy in the following sense: the étale groupoid associated with an AF inverse monoid under non-commutative Stone duality is the same as the groupoid associated with AF C^* -algebras.

6.1. Preliminaries. If P is a poset and $a \in P$, we write a^\downarrow for the set $\{b \in P: b \leq a\}$ and $a^\uparrow = \{b \in P: a \leq b\}$. A subset Q of P is called an *order ideal* if $q \in Q$ and $p \leq q$ implies that $p \in Q$. Let $D(S)$ be the inverse semigroup of all finitely generated compatible order ideals of S . This is the (*finitary*) *Schein completion* of S .

Proposition 6.1 (Schein completion). *The Schein completion $D(S)$ of an inverse semigroup S is a distributive inverse semigroup and the map $\sigma: S \rightarrow D(S)$, given by $s \mapsto s^\downarrow$, is universal for homomorphisms to distributive inverse semigroups.*

Let P be a poset with zero. We say that P is *unambiguous* if whenever $a, b, c \in P$ where $a \neq 0$ such that $a \leq b, c$ then $b \leq c$ or $c \leq b$. We say that P is *Dedekind finite* if for each non-zero element $a \in P$ the set a^\uparrow is finite. An inverse semigroup with zero is said to be *E^* -unitary* if $0 \neq e \leq a$ where e is an idempotent implies that a is an idempotent. The following is [33, Lemma 2.17].

Lemma 6.2. *Let S be an inverse monoid with zero which is E^* -unitary and whose semilattice of idempotents is unambiguous. Then the natural partial order on S is unambiguous.*

The following result can easily be proved directly, though a proof may be found in [33].

Lemma 6.3. *Let S be an inverse monoid with zero that has an unambiguous natural partial order. Then each finitely generated compatible order ideal of S can be generated by a finite set of pairwise orthogonal elements.*

Remark 6.4. We shall later construct the Schein completion $D(S)$ of an inverse monoid with an unambiguous order. By Lemma 6.3, we need only consider finitely generated compatible order ideals generated by orthogonal elements.

A congruence on a semigroup with zero is said to be *0-restricted* if the zero forms a congruence class on its own. A congruence is said to be *idempotent-pure* if the congruence class of an idempotent only contains idempotents.

6.2. Bratteli inverse monoids. We introduce a second inverse monoid constructed from a Bratteli diagram B that will ultimately shed light on the structure of $I(B)$.

A Bratteli diagram is simply a type of (rooted) directed graph, and from any (rooted) directed graph we may construct an inverse monoid in a way that seems first to have been employed in [5], but has been rediscovered many times. Let G be any directed graph. We denote by G^* the free category generated by G . The *graph inverse semigroup* P_G consists of a zero and all symbols xy^{-1} where x and y are elements of G^* that begin at the same vertex together with the following multiplication

$$xy^{-1} \cdot uv^{-1} = \begin{cases} xzv^{-1} & \text{if } u = yz \text{ for some path } z \\ x(vz)^{-1} & \text{if } y = uz \text{ for some path } z \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that we do indeed get an inverse semigroup in this way. The non-zero idempotents are the elements of the form xx^{-1} . The natural partial order is given by

$$xy^{-1} \leq uv^{-1} \Leftrightarrow \exists p \in \mathcal{G}^* \text{ such that } x = up \text{ and } y = vp.$$

An abstract characterization of graph inverse semigroups was given in [33]. A directed graph G is said to be *rooted* if there is a vertex v_0 , called the *root*, such that given any vertex v in G there is a path from v to v_0 . Let G be a rooted directed graph with root v_0 . Define P_G^\bullet to be the subset of P_G consisting of zero and all elements xy^{-1} where x and y both end at the root v_0 . Then, in fact, P_G^\bullet is a local submonoid of P_G and, though we shall not need this fact here, P_G is what is called an enlargement of P_G^\bullet [31]. We shall denote the identity of P_G^\bullet by 1. It is equal to $1_{v_0}1_{v_0}^{-1}$. Bratteli diagrams B are rooted directed graphs. Observe that to be concordant with our definitions above, you should think of the edges as being directed in the reverse direction in order that v_0 be a root, but this has little significance. We may therefore construct the graph inverse monoid P_B^\bullet . We call this the *Bratteli inverse monoid* constructed from the Bratteli diagram B . It is an obvious question to determine the relationship between P_B^\bullet and $\mathbf{l}(B)$ and this will be our main goal. We begin by determining some of the properties of P_B^\bullet . To do this, we shall need the following notion. Let S be an inverse semigroup. A function $\beta: S \setminus \{0\} \rightarrow \mathbb{N}$ is called a *weight function* if it satisfies the following axioms:

- (W1): $s < t$ implies that $\beta(s) > \beta(t)$.
- (W2): $s \mathcal{D} t$ implies that $\beta(s) = \beta(t)$.

Lemma 6.5. *Let S be an inverse monoid equipped with a weight function β .*

- (1) *S is completely semisimple.*
- (2) *If $E(S)$ is unambiguous and $st \neq 0$ then $\beta(st) = \max\{\beta(s), \beta(t)\}$.*

Proof. (1) Suppose that $e \mathcal{D} f \leq e$ for idempotents e, f . Then $\beta(e) = \beta(f)$. We are given that $f \leq e$. But the inequality cannot be strict and so $e = f$.

(2) Suppose that $st \neq 0$. Put $e = s^{-1}stt^{-1} \neq 0$. Then $st = (se)(te)$ where st and s and t are all \mathcal{D} -related. It follows that $\mu(st) = \mu(se)$. But $(se)^{-1}se = e$ and so $\mu(st) = \mu(e)$. Now $e = s^{-1}s \wedge tt^{-1}$. Thus by unambiguity, we have that $s^{-1}s \leq t^{-1}t$ or $t^{-1}t \leq s^{-1}s$. Suppose, without loss of generality, $s^{-1}s \leq t^{-1}t$. Then $\mu(st) = \mu(s^{-1}s) = \mu(s)$. But $\mu(s^{-1}s) \geq \mu(t^{-1}t)$ and so $\mu(s) \geq \mu(t)$. It follows that in this case $\mu(st) = \max\{\mu(s), \mu(t)\}$. \square

Part (4) below shows that Bratteli diagrams can be regarded as the posets of principal ideals of an inverse monoid. Note that an inverse semigroup S is *combinatorial* if $a, b \in S$ are such that $\mathbf{d}(a) = \mathbf{d}(b)$, and $\mathbf{r}(a) = \mathbf{r}(b)$, then $a = b$.

Proposition 6.6. *Let B be a Bratteli diagram.*

- (1) *The inverse monoid P_G^\bullet is equipped with a weight function such that $\mu^{-1}(0) = 1$ and for each $n \in \mathbb{N}$ the set $\mu^{-1}(n)$ is finite and non-empty. If $s < t$ then there exists $s \leq t' < t$ such that $\mu(t') = \mu(t) + 1$.*
- (2) *The inverse monoid P_G^\bullet is completely semisimple, combinatorial and E^* -unitary.*
- (3) *The semilattice of idempotents is unambiguous, above each non-zero idempotent are only a finite number of idempotents, and there are no atoms.*
- (4) *B , with a zero adjoined at the bottom, is the Hasse diagram of $P_B^\bullet / \mathcal{J}$.*

Proof. Most of these results are straightforward to prove. We simply highlight the key points.

If xy^{-1} is a nonzero element then x and y are paths that must begin at the same vertex of B and end at the root. It follows that x and y must also be the same

length. We define $\mu(xy^{-1}) = |x| = |y|$. It is clear that (W1) holds. The fact that (W2) holds follows from the fact that $xy^{-1} \mathcal{D} uv^{-1}$ if, and only if, y and v start at the same vertex.

Let xx^{-1} and yy^{-1} be non-zero idempotents. Then $xx^{-1} \mathcal{D} yy^{-1}$ if and only if x and y begin at the same vertex v . It follows that there is a bijection between the non-zero \mathcal{D} -classes and the vertices of the Bratteli diagram. Let v be a vertex at level n . Let P_v be the set of all strings that start at v and end at the root, remembering our convention about edge directions, here. The number of elements in P_v is just the size of v , defined earlier. The set of all elements xy^{-1} of P_B^\bullet where $x, y \in P_v$ forms a connected principal groupoid with $|P_v|$ identities. These elements also constitute a single \mathcal{D} -class of P_B^\bullet .

It is immediate that the semigroup is E^* -unitary and combinatorial.

There are no atoms. Let xx^{-1} be any non-zero idempotent. Then x is a path from the vertex v , at level n , to the root v_0 . Let e be an edge at level $n+1$ that ends in v . From the definition of a Bratteli diagram, such an edge e exists. Then $xe(xe)^{-1} \leq xx^{-1}$. \square

The following is immediate by the above. We state it explicitly since it will be important later.

Corollary 6.7. *The natural partial order of a Bratteli inverse monoid is unambiguous.*

Remark 6.8. It is possible to characterize Bratteli inverse monoids abstractly. We do not do this here.

6.3. Tight completions. We now have two inverse monoids associated with a Bratteli diagram B : the Bratteli inverse monoid P_B^\bullet and the AF inverse monoid $I(B)$. Our goal is to explain how they are related. We shall do this in the next section. Here, we describe the theory of essential completions of inverse semigroups. This is described in [45, 46] and is based on ideas that generalize constructions to be found in Section 5 of [47] as well as in [44] and are related to those to be found in [21, 22]. Our presentation here, though, is based on [45, 46] but we also take the opportunity to clarify some aspects of the theory developed there.

Let S be an inverse monoid with zero. Given elements $a, b \in S$ such that $b \leq a$, we say that b is *essential in a* or that b is *essentially contained in a* if for each $0 \neq x \leq a$, the meet $x \wedge b \neq 0$. The following motivates the definition.

Lemma 6.9. *Let S be a Boolean inverse monoid. Let $a, b \in S$ such that $b \leq a$. Then if b is essentially contained in a then $b = a$.*

Proof. We prove first that $b^{-1}b$ is essentially contained in $a^{-1}a$. Let $0 \neq e \leq a^{-1}a$. Then $ae \leq a$ and $ae \neq 0$. Thus $ae \wedge b \neq 0$. But since $ae, b \leq a$ they are compatible and so $\mathbf{d}(ae \wedge b) = \mathbf{d}(ae) \wedge \mathbf{d}(b)$. It follows that $e \wedge b^{-1}b \neq 0$, as required.

Now suppose that f is an idempotent essentially contained in e . Then $e = e \vee f\bar{e}$ and $f \wedge e\bar{f} = 0$. Thus $e\bar{f} = 0$ and so $e = f$.

Using this argument, we see that $b^{-1}b = a^{-1}a$ and so $b = a$, as claimed. \square

We now extend the definition from individual elements to finite subsets. A finite subset $\{a_1, \dots, a_m\} \subseteq a^\downarrow$ is said to be an *(essential) cover of a* if for each $0 \neq x \leq a$ we have that $x \wedge a_i \neq 0$ for some i . We shall write $A \preceq a$ to mean A is an (essential) cover of a . Since the only covers to be considered in this paper are essential ones we shall simply say *cover* from now on. The notions of an essential element and an essential subset are related.

Lemma 6.10.

- (1) Let S be a distributive inverse semigroup. Then $\{a_1, \dots, a_m\} \preceq a$ if and only if $\bigvee_{i=1}^m a_i \preceq a$.
- (2) Let S be an inverse semigroup with zero. Then $\{a_1, \dots, a_m\} \preceq a$ if, and only if, $\{a_1, \dots, a_m\}^\downarrow \preceq a^\downarrow$ in $D(S)$.

Proof. (1) Let $0 \neq x \leq a$. By part (3) of Lemma 2.2, we have that

$$x \wedge \left(\bigvee_{i=1}^m a_i \right) = \bigvee_{i=1}^m x \wedge a_i.$$

Suppose that $\{a_1, \dots, a_m\} \preceq a$. Then $x \wedge a_i \neq 0$ for some i . It follows that $x \wedge (\bigvee_{i=1}^m a_i) \neq 0$. Conversely, suppose that $x \wedge (\bigvee_{i=1}^m a_i) \neq 0$. Then $x \wedge a_i \neq 0$ for some i .

(2) Straightforward, □

The proofs of the following are all straightforward.

Lemma 6.11. *Let S be an inverse semigroup with zero.*

- (1) The relation \preceq is a partial order.
- (2) $b \preceq a$ implies that $b^{-1} \preceq a^{-1}$.
- (3) $b \preceq a$ and $d \preceq c$ implies that $bd \preceq ac$.
- (4) $0 \preceq a$ implies that $a = 0$.
- (5) If $b \preceq a$ then $\mathbf{d}(b) \preceq \mathbf{d}(a)$.
- (6) Let $b \leq a$. Then $b \preceq a$ if and only if $\mathbf{d}(b) \preceq \mathbf{d}(a)$.
- (7) Let $b, c \preceq a$. Then $b \wedge c \preceq a$.
- (8) Suppose that S is a \wedge -semigroup. If $a \preceq b$ and $c \preceq d$ then $a \wedge b \preceq b \wedge d$.
- (9) Suppose that S is a distributive inverse semigroup. If $a \preceq b$ and $c \preceq d$, and in addition $a \sim c$ and $b \sim d$ then $a \vee b \preceq b \vee d$.

If A is a subset of S then we define $\mathbf{d}(A) = \{\mathbf{d}(a) : a \in A\}$. We also have the following.

Lemma 6.12. *Let S be an inverse semigroup with zero.*

- (1) $\{a\} \preceq a$.
- (2) $A \preceq a$ implies that $A^{-1} \preceq a^{-1}$.
- (3) $A \preceq a$ and $B \preceq b$ imply that $AB \preceq ab$.
- (4) If $X \preceq a$ and $X_i \preceq x_i$ for each $x_i \in X$ then $\bigcup_i X_i \preceq a$. This is called transitivity of covers.
- (5) If $A \preceq a$ then $\mathbf{d}(A) \preceq \mathbf{d}(a)$.
- (6) Let $A \subseteq a^\downarrow$. Then $A \preceq a$ if and only if $\mathbf{d}(A) \preceq \mathbf{d}(a)$.
- (7) Let $A, B \preceq a$. Define $A \wedge B = \{a' \wedge b' : a' \in A, b' \in B\}$. Then $A \wedge B \preceq a$ and $A \wedge B = \mathbf{Ad}(B) = \mathbf{Bd}(A)$.
- (8) Let S be an inverse \wedge -semigroup. If $A \preceq a$ and $B \preceq b$ then $A \wedge B \preceq a \wedge b$.

The following was introduced in [47]. Let S be an arbitrary inverse semigroup with zero. Let $a \in S$ and let $\{a_1, \dots, a_m\}$ be a non-empty finite subset. We write $a \rightarrow \{a_1, \dots, a_m\}$ if for each $0 \neq x \leq a$ we have that $x^\downarrow \cap a_i^\downarrow \neq 0$ for some i . In the case of $a \rightarrow \{b\}$, we simply write $a \rightarrow b$. We write $\{a_1, \dots, a_m\} \rightarrow \{b_1, \dots, b_n\}$ iff $a_i \rightarrow \{b_1, \dots, b_n\}$ for $1 \leq i \leq m$. We write $\{a_1, \dots, a_m\} \leftrightarrow \{b_1, \dots, b_n\}$ iff $\{a_1, \dots, a_m\} \rightarrow \{b_1, \dots, b_n\}$ and $\{b_1, \dots, b_n\} \rightarrow \{a_1, \dots, a_m\}$. The proof of the following is immediate.

Lemma 6.13. *Let $\{a_1, \dots, a_m\} \subseteq a^\downarrow$. Then $\{a_1, \dots, a_m\} \preceq a$ if, and only if, $\{a_1, \dots, a_m\} \rightarrow a$.*

We now come to the key definition. A homomorphism $\theta: S \rightarrow T$ from an inverse semigroup S to a distributive inverse semigroup T is said to be *tight* if for each

$a \in S$ and $A \preceq a$ we have that $\theta(a) = \bigvee_{a_i \in A} \theta(a_i)$. Thus a tight homomorphism converts covers to joins. A *tight completion* of S is a distributive inverse semigroup $D_t(S)$ together with a tight homomorphism $\tau: S \rightarrow D_t(S)$ which is universal. If such a completion exists then it is, of course, unique up to isomorphism. We shall show that the essential completion exists.

Remark 6.14. Let S be a distributive inverse semigroup. If $a = \bigvee_{i=1}^m a_i$ then $\{a_1, \dots, a_m\}$ is a cover of a . It follows that tight maps between distributive inverse semigroups preserve any finite joins that exist. Thus they are morphisms of distributive inverse semigroups.

A morphism $\theta: S \rightarrow T$ between distributive inverse semigroups is said to be *essential* if $x \preceq s$ implies that $\theta(x) = \theta(s)$.

Lemma 6.15. *A morphism $\theta: S \rightarrow T$ between distributive inverse semigroups is essential if and only if it is tight.*

Proof. Suppose that θ is essential. Let $\{a_1, \dots, a_m\} \subseteq a^\downarrow$ be a cover. Then $b = \bigvee_{i=1}^m a_i$ is essential in a by Lemma 6.10. By assumption $\theta(a) = \theta(b)$. We now use the fact that θ is a morphism and so $\theta(b) = \bigvee_{i=1}^m \theta(a_i)$ which gives $\theta(a) = \bigvee_{i=1}^m \theta(a_i)$, as required. The proof of the converse is immediate. \square

It follows that in the case of distributive inverse monoids the word *tight* may be replaced by the word *essential*.

We begin by constructing the essential completion of a distributive inverse semigroup. Let S be a distributive inverse monoid. Define the relation \equiv on S as follows

$$a \equiv b \iff z \preceq a, b$$

for some $z \in S$. We prove below that this is a congruence. The \equiv -class containing a is denoted by $[a]$.

Lemma 6.16. *Let S be a distributive inverse semigroup. The relation \equiv is a 0-restricted congruence on S . If S is \wedge -semigroup so is S/\equiv .*

Proof. The fact that the relation is an equivalence relation, a congruence and 0-restricted all follow from Lemma 6.11.

It remains to prove that if S is an inverse \wedge -semigroup so is S/\equiv . Consider the elements $[a]$ and $[b]$. By assumption, $a \wedge b$ exists and $[a \wedge b] \leq [a], [b]$. Now let $[z] \leq [a], [b]$. Then $z \equiv az^{-1}z \equiv bz^{-1}z$. Let $u \preceq z, az^{-1}z$ and $v \preceq z, bz^{-1}z$. Then $u \wedge v \preceq z, az^{-1}z, bz^{-1}z$ by Lemma 6.11. It follows that $[z] = [az^{-1}z \wedge bz^{-1}z] \leq [a \wedge b]$, as required. \square

Proposition 6.17. *Let S be a distributive inverse semigroup. Suppose that $e \preceq a$, where e is an idempotent, implies that a is an idempotent. Then \equiv is idempotent-pure and S/\equiv is the essential completion of S .*

Proof. It is immediate from our assumption that \equiv is idempotent-pure. Put $U = S/\equiv$. We show first that U is a distributive inverse semigroup. Suppose that $[a] \sim [b]$. Then both ab^{-1} and $a^{-1}b$ are idempotents because \equiv is idempotent-pure and so $a \sim b$. It follows that $a \vee b$ exists in S . Since $a, b \leq a \vee b$ we have that $[a], [b] \leq [a \vee b]$. Let $[a], [b] \leq [c]$. Then $a \equiv ca^{-1}a$ and $b \equiv cb^{-1}b$. By definition, we have that for some x we have $x \preceq a$ and $x \preceq ca^{-1}a$. Similarly, we have that for some y we have that $y \preceq b$ and $y \preceq cb^{-1}b$. It follows that $x \sim y$ since $x, y \leq a \vee b$. Thus by Lemma 6.11, $x \vee y \preceq a \vee b$ and $x \vee y \preceq c$. Hence $[a \vee b] \leq [c]$, as required. It is now straightforward to check that U is a distributive inverse semigroup.

To prove that $\nu: S \rightarrow U$ is an essential map, observe that $b \preceq a$ implies that $b \Leftrightarrow a$ and so $[b] = [a]$.

Let $\theta: S \rightarrow T$ be an essential homomorphism to a distributive inverse semigroup. Denote the natural map from S to S/\equiv by ν . Define

$$\bar{\theta}: S/\equiv \longrightarrow T \text{ by } \bar{\theta}([a]) = \theta(a).$$

We show first that $\bar{\theta}$ is well-defined. Suppose that $[a] = [b]$. Then $a \equiv b$. It follows that there is an element x such that $x \preceq a$ and $x \preceq b$. By assumption, $\theta(x) = \theta(a)$ and $\theta(x) = \theta(b)$. It follows that $\theta(a) = \theta(b)$. We need to prove that $\bar{\theta}$ preserves any binary joins that exist. But this we have essentially done above. By construction, we have that $\bar{\theta}\nu = \theta$ and it is immediate that $\bar{\theta}$ is unique with these properties. \square

We now describe how to construct the tight completion.

Proposition 6.18. *Let S be an inverse semigroup. Let $\theta: S \rightarrow T$ be a tight homomorphism to a distributive inverse semigroup. Then the unique morphism $\theta^*: D(S) \rightarrow T$ such that $\theta^*\iota = \theta$ is an essential morphism.*

Proof. It is enough to prove that θ^* is an essential map. Let $A = \{a_1, \dots, a_m\}^\downarrow$ and $B = \{b_1, \dots, b_n\}^\downarrow$ be two elements of $D(S)$ such that $B \preceq A$ in $D(S)$. We shall prove that $\theta^*(A) = \theta^*(B)$. By definition $\theta^*(B) = \bigvee_{i=1}^n \theta(b_i)$ and $\theta^*(A) = \bigvee_{j=1}^m \theta(a_j)$. Clearly $\theta^*(B) \leq \theta^*(A)$. Thus by definition, we have that $\theta^*(B) = \theta^*(A)(\bigvee_{j=1}^m \theta(\mathbf{d}(b_j)))$. We shall prove that $\theta(a_i) = \theta(a_i)(\bigvee_{j=1}^m \theta(\mathbf{d}(b_j)))$ from which the result follows and to do that it is enough to prove that $a_i \rightarrow \{a_i \mathbf{d}(b_1), \dots, a_i \mathbf{d}(b_n)\}$. Let $0 \neq z \leq a_i$. Then $0 \neq z^\downarrow \leq A$. It follows that there is a non-zero $C \in D(S)$ such that $C \leq z^\downarrow, B$. We may therefore find $0 \neq c \in C$ such that $c \leq z, b_j$ for some j . It follows that $0 \neq c \leq a_i \mathbf{d}(b_j), b_j$. \square

Let S be an arbitrary inverse semigroup with zero and let $a, b \in S$. Define $a \cong b$ if and only if $A \preceq a, b$ for some cover A . We say that a and b have a common cover.

Lemma 6.19. *The relation \cong is a 0-restricted congruence. If $a, b \leq c$ then $[a \wedge b] = [a] \wedge [b]$.*

Proof. The proof of the main claim follows from Lemma 6.12. We now prove the second claim. Since $a, b \leq c$ we have that $[a], [b] \leq [c]$ and so $[a] \wedge [b]$ exists. Clearly, $[a \wedge b] \leq [a] \wedge [b]$. Let $[z] \leq [a], [b]$. We have that $z \cong az^{-1}z \equiv bz^{-1}z$. It follows that $z \cong (a \wedge b)z^{-1}z$, as required. \square

We denote the \cong -class containing a by $[a]$. We say that an inverse semigroup is *separative* if \cong is the equality relation.

Lemma 6.20. *Let S be an inverse semigroup. Then S/\cong is separative.*

Proof. Suppose that $[a] \cong [b]$. Let $\{[x_1], \dots, [x_m]\} \preceq [a], [b]$. Put $e_i = \mathbf{d}(x_i)$ for $1 \leq i \leq m$. Then $\{[e_1], \dots, [e_m]\} \preceq [\mathbf{d}(a)], [\mathbf{d}(b)]$. Observe that $ae_1, \dots, ae_m \leq a$ and that $ae_i \cong x_i$ for $1 \leq i \leq m$. It is easy to check that $A = \{ae_1, \dots, ae_m\} \preceq a$. By a similar argument, $B = \{be_1, \dots, be_m\} \preceq b$. Now $ae_i \cong be_i$. Let $C_i = \{c_{i1}, \dots, c_{in}\}$ be a common cover of ae_i and be_i . We are interested in the set $\{c_{ij}\}$. It is a subset of both a^\downarrow and b^\downarrow . Now A is a cover of a and for each element $ae_i \in A$ we have that C_i is a cover of ae_i . It follows by ‘transitivity of covers’ that C is a cover of a . By symmetry, C is a cover of b . It follows that a and b have a common cover and so $a \cong b$. \square

We denote by $\mu: S \rightarrow S/\cong$ the natural map. The following is proved in [46].

Proposition 6.21. *Let S be an inverse semigroup and let $\theta: S \rightarrow T$ be a tight homomorphism to a distributive inverse semigroup T . Then there is a unique tight homomorphism $\bar{\theta}: S/\cong \rightarrow T$ such that $\bar{\theta}\mu = \theta$.*

By the lemma above, we need only construct the tight completion in the separative case. The lemma below will provide the connection with idempotent-purity.

Lemma 6.22. *Let S be a separative inverse semigroup. Let $A \preceq a$ where all elements of A are idempotents. Then a is an idempotent.*

Proof. Observe that $\mathbf{d}(A) = A$ and so $A \preceq \mathbf{d}(a)$. It follows that $a \cong \mathbf{d}(a)$. Thus $a = \mathbf{d}(a)$ by separativity. \square

Lemma 6.23. *Let S be an inverse semigroup and $A = \{a_1, \dots, a_m\}^\downarrow \subseteq B = \{b_1, \dots, b_n\}^\downarrow$ both be elements of $\mathbf{D}(S)$. Then $A \preceq B$ if and only if $\{b_1, \dots, b_n\} \rightarrow \{a_1, \dots, a_m\}$.*

Proof. Suppose first that $A \preceq B$. Let $0 \neq x \leq b_i$. Then $0 \neq x^\downarrow \leq B$. By assumption there exists $0 \neq C \leq x^\downarrow, A$. Let $0 \neq c \in C$. Then $c \leq x, a_j$ for some j . To prove the converse, suppose that $\{b_1, \dots, b_n\} \rightarrow \{a_1, \dots, a_m\}$. Let $0 \neq C \leq B$ where $C = \{c_1, \dots, c_p\}^\downarrow$. By assumption, for each k we may find x_k such that $0 \neq x_k \leq c_k, a_{i_k}$. Put $X = \{x_1, \dots, x_p\}^\downarrow$. Then $X \neq 0$, $X \leq C$ and $X \leq A$. \square

We now have the following.

Lemma 6.24. *Let S be a separative inverse semigroup. Then the congruence \equiv is idempotent-pure on $\mathbf{D}(S)$.*

Proof. Let $A = \{a_1, \dots, a_m\}^\downarrow$ and $B = \{b_1, \dots, b_n\}^\downarrow$. Then $A \equiv B$ if and only if there exists $C \in \mathbf{D}(S)$ such that $C \preceq A$ and $C \preceq B$. Suppose that B is an idempotent. Then C is an idempotent. But $C \preceq A$. Let $C = \{e_1, \dots, e_p\}^\downarrow$ where the e_i are idempotents. We use the fact that C is a compatible order ideal. It follows that for each a_j we have that $C \wedge a_j^\downarrow$ is defined and that $C \wedge a_j^\downarrow \preceq a_j^\downarrow$. But all the elements of $C \wedge a_j^\downarrow$ are idempotents and so by Lemma 6.22, we have that a_j is an idempotent. Thus A is an idempotent, as required. \square

We now have all the elements needed for our first main theorem.

Theorem 6.25 (Tight completion). *Let S be an inverse semigroup with zero. Let \mathbf{S} be its separative image. Then $\mathbf{D}_t(S) = \mathbf{D}(\mathbf{S})/\equiv$.*

Proof. Let $\theta: S \rightarrow T$ be a tight map to a distributive inverse semigroup. Then by Proposition 6.21, we may replace S by its separative image \mathbf{S} . By Proposition 6.18, we may replace \mathbf{S} by its Schein completion $\mathbf{D}(\mathbf{S})$. But by Lemma 6.24, the congruence \equiv is idempotent-pure on $\mathbf{D}(\mathbf{S})$. We may now apply Proposition 6.17 to get the result. \square

The following will apply to the application of tight completions that will interest us.

Lemma 6.26. *Let S be an inverse monoid with zero.*

- (1) *If S is a \wedge -monoid then $\{a_1, \dots, a_m\}^\downarrow \equiv \{b_1, \dots, b_n\}^\downarrow$ if, and only if, $\{a_1, \dots, a_m\} \leftrightarrow \{b_1, \dots, b_n\}$.*
- (2) *If S is E^* -unitary, then \equiv is idempotent-pure on $\mathbf{D}(S)$.*

Proof. (1) Suppose that $A = \{a_1, \dots, a_m\}^\downarrow \equiv \{b_1, \dots, b_n\}^\downarrow = B$. Then there exists $C = \{c_1, \dots, c_p\}^\downarrow$ such that $\{c_1, \dots, c_p\}^\downarrow \preceq \{a_1, \dots, a_m\}^\downarrow$ and $\{c_1, \dots, c_p\}^\downarrow \preceq \{b_1, \dots, b_n\}^\downarrow$. Let $0 \neq x \leq a_i$. Then $0 \neq x^\downarrow \leq A$. Thus $x^\downarrow \wedge C \neq 0$. It follows that $x \wedge c_k \neq 0$ for some k . But $c_k \leq b_j$ for some j . Thus $x \wedge b_j \neq 0$ for some j . By symmetry, it follows that $\{a_1, \dots, a_m\} \leftrightarrow \{b_1, \dots, b_n\}$.

Conversely, suppose that $\{a_1, \dots, a_m\} \leftrightarrow \{b_1, \dots, b_n\}$. Put $C = \{a_i \wedge b_j : 1 \leq i \leq m, 1 \leq j \leq n\}^\downarrow$. It is now easy to check that $C \preceq A, B$. It follows that $A \equiv B$, as required.

(2) Let $A, B \in D(S)$ be non-zero elements such that A is an idempotent and $A \equiv B$. We prove that B is an idempotent. By definition, there is C such that $C \preceq A, B$. Since $C \subseteq A$, all the elements of C are idempotents. Let b be one of the generators of B . Then $b^\downarrow \leq B$. Thus b^\downarrow has a non-zero intersection with some element e^\downarrow where $e \in C$. It follows that b is above a non-zero idempotent and so itself an idempotent, as required. \square

6.4. Tight completions of Bratteli inverse monoids are AF. The goal of this section is to prove the following.

Theorem 6.27. *Let B be a Bratteli diagram. Then $\mathbf{l}(B)$ is the tight completion of P_B^\bullet .*

Since P_B^\bullet is E^* -unitary, it follows by Lemma 6.26, that we need to show $\mathbf{l}(B)$ is isomorphic to $D(P_B^\bullet)/\equiv$. By Lemma 6.3, Proposition 6.6 and Lemma 6.2, the inverse monoid P_B^\bullet has an unambiguous natural partial order and so every finitely generated compatible order ideal is generated by pairwise orthogonal elements.

We now describe, informally, why $D(P_B^\bullet)/\equiv$ is isomorphic to $\mathbf{l}(B)$. Let v be a vertex of the Bratteli diagram B . Denote by P_v all paths from v to the root. Denote by G_v the groupoid of all elements xy^{-1} where $x, y \in P_v$. These are just the non-zero \mathcal{D} -classes and so P_B^\bullet is a disjoint union of them. Think of these groupoids as attached to the appropriate vertices of the Bratteli diagram. Elements at one level can only be above, in the natural partial order, elements at lower levels, this order being mediated by edges since $xx^{-1} \leq yy^{-1}$ if, and only if, $x = yp$ for some path p . The key point is that if we adjoin a zero to G_v and take its distributive completion, we get a symmetric inverse monoid with letters the set P_v . In particular, a symmetric inverse monoid of the correct size associated with the vertex v of the Bratteli diagram B . The disjoint union of groupoids at one level is therefore inflated to a direct product of symmetric inverse monoids of the correct sizes for that level of the Bratteli diagram. The standard maps between adjacent levels are induced by the natural partial order linking elements at one level to the elements immediately below it. The elements lower down are essential in the elements immediately above and so are identified in the tight completion. It remains to convert this informal description into a detailed proof.

An element of $D(P_B^\bullet)$ is said to be *(p-)homogeneous* if it is generated by pairwise orthogonal elements that all have the same weight p . Define $D^h(P_B^\bullet)$ to be the subset of D consisting of all homogeneous elements of D and zero. If two elements of weight p are multiplied together the result is either zero or an element of weight p . It follows that $D^h(P_B^\bullet)$ forms an inverse submonoid of $D(P_B^\bullet)$ that we call the *homogeneous orthogonal completion*.

We now show how (tight) covers of idempotents in P_B^\bullet may arise.

Lemma 6.28 (Lengthening). *Let x be a path from the vertex v , at level n , to the root v_0 . Let e_1, \dots, e_q be all the edges at level $n+1$ that end in v . Then the set $A = \{xe_1(xe_1)^{-1}, \dots, xe_q(xe_q)^{-1}\}$ is a cover of xx^{-1} , and all the elements of A have length $n+1$.*

Proof. By the definition of a Bratteli diagram, there is at least one such edge. Any non-zero idempotent $yy^{-1} \leq xx^{-1}$ must be such that $y = xe_ip$ for some edge e_i and path p . Thus $yy^{-1} \leq xe_i(xe_i)^{-1}$. \square

The above process may be iterated. Observe that, from the definition of a Bratteli diagram, a path of length p can always be lengthened to a path of length $p+1$. The proof of the following is now almost immediate.

Lemma 6.29 (Homogenizing). *Every inhomogeneous element of $D(P_B^\bullet)$ is \equiv -related to a homogeneous one.*

Lemma 6.30. *Let A and B be two p -homogeneous elements of $D(P_B^\bullet)$. Then $A \equiv B$ implies that $A = B$.*

Proof. The result follows from the following observations. We are working in an inverse monoid with an unambiguous natural partial order. Two elements which have a non-zero lower bound must be comparable. But comparable elements of the same weight must be equal. \square

Thus the structure of $D(P_B^\bullet)/\equiv$ will be strongly influenced by the structure of $D^h(P_B^\bullet)$. In addition, the congruence \equiv cannot identify two p -homogeneous elements.

Consider now level p of the Bratteli diagram B . Let the vertices be v_1, \dots, v_k . Let the sizes of these vertices be m_1, \dots, m_k , respectively. Then the p -homogeneous elements are in bijective correspondence with the non-zero elements of $I_{m_1} \times \dots \times I_{m_k}$. This is because the distributive completion of the groupoid G_{v_j} with a zero adjoined is just I_{m_j} . Thus the p -homogeneous elements of $D^h(P_B^\bullet)$ with an adjoined zero form the *correct* semisimple inverse monoid. We now describe how successive levels are related.

For convenience, we put $S = P_B^\bullet$. Let $s \in S$ be any non-zero element. Define

$$\varepsilon(s) = \{t \in S : t \leq s \text{ and } \beta(t) = \beta(s) + 1\}.$$

This set consists of all elements *immediately below* s . This is a finite set since $\mu^{-1}(p)$ is finite for any p . This is a compatible subset because all elements are bounded above by s . This is an orthogonal set since two elements which are of the same weight and compatible are either equal or orthogonal. Define $\varepsilon(0) = \{0\}$.

Lemma 6.31.

- (1) $\varepsilon(st)^\downarrow = \varepsilon(s)^\downarrow \varepsilon(t)^\downarrow$.
- (2) $\varepsilon(s \wedge t)^\downarrow = \varepsilon(s)^\downarrow \wedge \varepsilon(t)^\downarrow$.
- (3) $\varepsilon(s)^\downarrow \preceq s^\downarrow$.

Proof. (1) We suppose that $st \neq 0$ and that $\beta(s) \geq \beta(t)$, without loss of generality. Clearly, $\varepsilon(s)^\downarrow \varepsilon(t)^\downarrow \subseteq \varepsilon(st)^\downarrow$ since multiplying two elements together is never weight-decreasing. Let $a \leq st$ of weight at least $\beta(s) + 1$. Then $0 \neq at^{-1} \leq stt^{-1} = s$. Thus $at^{-1} \in \varepsilon(s)^\downarrow$. Hence $a \in \varepsilon(s)^\downarrow (s^{-1}st)$ and $s^{-1}st \in \varepsilon(t)^\downarrow$.

(2) This is immediate.

(3) immediate. \square

Define $\eta(s) = \varepsilon(s)^\downarrow$. Put $\mathbf{e}_1 = \varepsilon(1)^\downarrow$. Then we have defined a morphism $\eta: S \rightarrow \mathbf{e}_1 D(S) \mathbf{e}_1$. Clearly, $e D(S) e$ is a distributive inverse monoid and so the map extends uniquely to a map $\eta: D(S) \rightarrow \mathbf{e}_1 D(S) \mathbf{e}_1$. This map is injective essentially by unambiguity. We therefore obtain a strictly decreasing sequence of idempotents $\mathbf{e}_1 > \mathbf{e}_2 > \mathbf{e}_3 > \dots$. It follows that, by restriction, we get injective morphisms $\eta_p: \mathbf{e}_p D(S) \mathbf{e}_p \rightarrow \mathbf{e}_{p+1} D(S) \mathbf{e}_{p+1}$ between local submonoids. The elements of maximum weight in $\mathbf{e}_p D(S) \mathbf{e}_p$ consist of the p -homogeneous elements of $D(S)$. Denote these by D_p . It follows that we get a map $\varepsilon_p: D_p \rightarrow D_{p+1}$ by restriction. We now calculate what this map does. Let the vertices at level p be v_1^p, \dots, v_k^p . Let $(\mathbf{s}_1, \dots, \mathbf{s}_k)$ be a k -tuple of p -homogeneous elements. Where $\mathbf{s}_j \in G_{v_j^p}$. We calculate the i th term of the image of this element under ε_p . There are s_{ij} edges joining vertex v_j^p and vertex v_i^{p+1} . We denote these edges by e_{ij}' .

We now make a simple observation. If $xy^{-1} \in G_v$ and e is an edge that connects v and v' , where v' is a vertex at level $p+1$. Then $(xe)(ye)^{-1} \leq xy^{-1}$ and belongs to $G_{v'}$. We say that $(xe)(ye)^{-1}$ is obtained from xy^{-1} by an *edge-adjunction using the edge e* .

It follows that the effect of ε_p is to carry out all possible edge-adjunctions on the given k -tuple of elements of weight p using all the edges joining level p to level $p+1$. It follows that on non-zero elements the effect of ε_p is that of the corresponding standard map. The proof of the following is now almost immediate.

Proposition 6.32. *Let B be a Bratteli diagram and let $S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \dots$ be the associated sequence of semisimple monoids and injective morphisms. Construct from this the ω -chain of inverse monoids S and factor out by the ideal \mathcal{Z} . Then the homogeneous orthogonal completion of P_B^\bullet is isomorphic to S/\mathcal{Z} .*

We now describe the effect of glueing elements together under \equiv . By Lemma 6.9, this relation is trivial on Boolean inverse monoids. It follows that it can only be non-trivial between the layers of $D^h(P_B^\bullet)$. By Lemma 6.31, if one element is mapped to another by ε_p then they will be identified by \equiv and the congruence \equiv is generated by these identifications.

6.5. The associated groupoid. From the theory described in [44], with each Boolean inverse \wedge -monoid we may associate a Hausdorff étale topological groupoid under non-commutative Stone duality. We shall describe the groupoid associated with $I(B)$. This looks like a difficult question. However, we have proved that this inverse monoid is constructed from P_B^\bullet . Again, by the theory described in [44], we will get the same groupoid if we start from the much simpler monoid P_B^\bullet . The elements of the groupoid are the ultrafilters in P_B^\bullet . Since this inverse monoid is E^* -unitary, the identities of the groupoid correspond to the idempotent ultrafilters and these are in bijective correspondence with the infinite paths in the Bratteli diagram B that end at the root. The non-identity elements of the groupoid are essentially cosets, and may be identified with triples (yw, yx^{-1}, xw) where w is an infinite path in B to the root and x and y are two paths that start from the same vertex and end at the root. It follows that the groupoid is just *tail-equivalence* [23].

REFERENCES

- [1] E. Akin, Measures on Cantor space, *Topology Proceedings* **24** (1999), 1–34.
- [2] E. Akin, Good measures on Cantor space, *Trans Amer. Math. Soc.* **357** (2004), 2681–2722.
- [3] E. Akin, R. Dougherty, R. D. Mauldin, A. Yingst, Which Bernoulli measures are good measures?, Preprint, 2005.
- [4] P. Ara, The realization problem for von Neumann regular rings, arXiv:0802.1872v1.
- [5] C. J. Ash, T. E. Hall, Inverse semigroups on graphs, *Semigroup Forum* **11** (1975), 140–145.
- [6] R. Baer, Sums of groups and their generalizations. An analysis of the associative law, *Amer. J. Math.* **71** (1949), 706–742.
- [7] E. Behrends, *Maß- und Integrationstheorie*, Springer-Verlag, 1987.
- [8] M. K. Bennett, D. J. Foulis, Phi-symmetric effect algebras, *Foundations of Physics* **25** (1995), 1699–1722.
- [9] V. Berthé, M. Rigo, Combinatorics on Bratteli diagrams and dynamical systems, in *Combinatorics, automata and number theory* (eds V. Berthé, M. Rigo), CUP, (2010), 338–386.
- [10] S. Bezuglyi, D. Handelman, Measures on Cantor sets: the good, the ugly, the bad, arXiv:1201.1953v1, 2012.
- [11] J.-C. Birget, The groups of Richard Thompson and complexity, *IJAC* **14** (2004), 569–626.
- [12] O. Bratteli, Inductive limits of finite dimensional C^* -algebras, *Trans Amer. Math. Soc.* **171** (1972), 195–234.
- [13] O. Bratteli, P. E. T. Jorgensen, *Iterated function systems and permutation representations of the Cuntz algebra*, AMS, (1999),
- [14] T. Ceccherini-Silberstein, R. Grigorchuk, P. de la Harpe, Amenability and paradoxical decompositions for pseudogroups and for discrete metric spaces, *Proc. Steklov Inst. Math.* **224** (1999), 57–97.
- [15] C. C. Chang, Algebraic analysis of many valued logic, *Trans Amer. Math. Soc.* **88** (1958), 467–490.
- [16] R. Cignoli, I. M. L. D’Ottaviano, D. Mundici, *Algebraic foundations of many-valued reasoning*, Trends in Logic, Vol. 7, Kluwer, Dordrecht, 2000.

- [17] A. Dudko, K. Medynets, On characters of inductive limits of symmetric groups, *ESI* 31st May, 2011.
- [18] A. Dvurečenskij, S. Pulmannová, *New trends on quantum structures*, Kluwer Acad. Publ. Dordrecht/Boston/London and Ister Science, Bratislava, (2000).
- [19] E. G. Effros, *Dimensions and C^* -algebras*, AMS, 1981.
- [20] G. A. Elliott, On the classification of inductive limits of sequences of semisimple finite-dimensional algebras, *J. A.* **38** (1976), 29–44.
- [21] R. Exel, Inverse semigroups and combinatorial C^* -algebras, *Bull. Braz. Math. Soc. (N.S.)* **39** (2008), 191–313.
- [22] R. Exel, Tight representations of semilattices and inverse semigroups, *Semigroup forum* **79** (2009), 159–182.
- [23] R. Exel, J. Renault, AF-algebras and the tail-equivalence relation, *Proc. A. M. S.* **134** (2006), 193–206.
- [24] D. J. Foulis, MV and Heyting effect algebras, *Foundations of Physics* **30** (2000), 1687–1706.
- [25] D. J. Foulis, M. K. Bennett, Effect algebras and unsharp quantum logics, *Foundations of Physics* **24** (1994), 1331–1352.
- [26] K. R. Goodearl, *Notes on real and complex C^* -algebras*, Shiva Publishing Limited, 1982.
- [27] K. R. Goodearl, F. Wehrung, *The complete dimension theory of partially ordered systems with equivalence and orthogonality*, *Memoirs A. M. S.* **176**, (2005).
- [28] B. Jacobs, New directions in categorical logic, for classical, probabilistic, and quantum logic, arXiv: 1205.3940v3, June, 2014.
- [29] G. Jenča, Boolean algebras R -generated by MV-effect algebras, *Fuzzy Sets and Systems* **145** (2004), 279–285.
- [30] A. Jenčová, S. Pulmannová, A note on effect algebras and dimension theory of AF C^* -algebras, *Reports on Mathematical Physics* **62** (2008), 205–218.
- [31] D. G. Jones, Polycyclic monoids and their generalisations, PhD Thesis, Heriot-Watt University, 2011.
- [32] D. G. Jones, M. V. Lawson, Strong representation of the polycyclic inverse monoids: cycles and atoms, *Periodica Math. Hung.* **64** (2012), 53–87.
- [33] D. G. Jones, M. V. Lawson, Graph inverse semigroups: their characterization and completion, accepted by *J. Alg.*
- [34] J. Kellendonk, The local structure of tilings and their integer group of coinvariants, *Comm. Math. Phys.* **187** (1997), 115–157.
- [35] N. K. Kroshko, V. I. Sushchansky, Direct limits of symmetric and alternating groups with strictly diagonal embeddings, *Archiv Math.* **71** (1998), 173–182.
- [36] A. Kumjian, On localizations and simple C^* -algebras, *Pacific J. Math.* **112** (1984), 141–192.
- [37] Y. Lavrenyuk, V. Nekrashevych, On classification of inductive limits of direct products of alternating groups, *J. London Math. Soc.* **75** (2007), 146–162.
- [38] M. V. Lawson, *Inverse semigroups: the theory of partial symmetries*, World Scientific, 1998.
- [39] M. V. Lawson, Orthogonal completions of the polycyclic monoids, *Comms Alg.* **35** (2007), 1651–1660.
- [40] M. V. Lawson, The polycyclic monoids P_n and the Thompson groups $V_{n,1}$, *Comms Alg.* **35** (2007), 4068–4087.
- [41] M. V. Lawson, Primitive partial permutation representations of the polycyclic monoids and branching function systems, *Periodica Math. Hung.* **52** (2009), 189–207.
- [42] M. V. Lawson, A non-commutative generalization of Stone duality, *J. Aust. Math. Soc.* **88** (2010), 385–404.
- [43] M. V. Lawson, Compactable semilattices, *Semigroup Forum* **81** (2010), 187–199.
- [44] M. V. Lawson, Non-commutative Stone duality: inverse semigroups, topological groupoids and C^* -algebras, *IJAC* **22**, 1250058 (2012) DOI:10.1142/S0218196712500580.
- [45] M. V. Lawson, D. H. Lenz, Pseudogroups and their étale groupoids, *Advances in Maths* **244** (2013), 117–170.
- [46] M. V. Lawson, D. H. Lenz, Distributive inverse semigroups and non-commutative Stone dualities, preprint, arXiv:1302.3032v1.
- [47] D. H. Lenz, On an order-based construction of a topological groupoid from an inverse semigroup, *Proc. Edinb. Math. Soc.* **51** (2008), 387–406.
- [48] J. Meakin, The partially ordered set of \mathcal{J} -classes of a semigroup, *J. Lond. Math. Soc. (2)*, **21** (1980), 244–256.
- [49] J. Meakin, M. Sapir, Congruences on free monoids and submonoids of polycyclic monoids, *J. Austral. Math. Soc. (Series A)* **54** (1993), 236–253.
- [50] D. Mundici, Interpretation of AF C^* -algebras in Łukasiewicz sentential calculus, *J. Func. Analysis* **65** (1986), 15–63.
- [51] D. Mundici, Logic of infinite quantum systems, *I. J. Theor. Phys.* **32** (1993), 1941–1955.

- [52] D. Mundici, MV-algebras, May 26th, 2007,
http://www.matematica.uns.edu.ar/IXCongresoMonteiro/Comunicaciones/Mundici_tutorial.pdf
- [53] D. Mundici, G. Panti, Extending addition in Elliot's local semigroup, *J. Func. Analysis* **117** (1993), 461–472.
- [54] J. v. Neumann, Continuous Geometries, *Proc. Nat. Acad. Sciences (USA)*, Vol. 22 (1936), 92–100.
- [55] J. v. Neumann, Examples of Continuous Geometries, *Proc. Nat. Acad. Sciences (USA)* Vol. 22 (1936), 101–108.
- [56] D. Perrin, J.-E. Pin, *Infinite words*, Elsevier, 2004.
- [57] J. R. Peters, R. J. Zerr, Partial dynamical systems and AF C^* -algebras, arXiv:math/0301337v1.
- [58] S. Pulmannová, Effect algebras with the Riesz decomposition property and AF C^* -algebras, *Foundations of Physics* **29** (1999), 1389–1401.
- [59] K. Ravindran, *On a structure theory of effect algebras*, PhD dissertation, Kansas State University, Manhattan, Kansas, 1996.
- [60] J. Renault, *A groupoid approach to C^* -algebras*, Lecture Notes in Mathematics, **793**, Springer, 1980.
- [61] L. Solomon, Representations of the rook monoid, *J. Alg.* **256** (2002), 309–342.
- [62] A. R. Wallis, *Semigroup and category-theoretic approaches to partial symmetry*, PhD Thesis, 2013, Heriot-Watt University, Edinburgh, UK.
- [63] F. Wehrung, The dimension monoid of a lattice, *Alg. Univ.* **40** (1998), 247–411.

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